Open University Course Units 0-6

M334 Differential Geometry

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M334 PART 0

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

PART 0 GUIDE TO THE COURSE

DIFFERENTIAL GEOMETRY

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

DIFFERENTIAL GEOMETRY PART 0

O.U. COURSE UNITS COLLECTION
FOR REFERENCE ONLY

GUIDE TO THE COURSE

Prepared by the Course Team

THE OPEN UNIVERSITY PRESS

M334 0

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LOAN CAT NL

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This text forms part of the correspondence element of an Open University Third Level Course. The complete list of parts in the course is given at the end of this text.

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Further information on Open University courses may be obtained from The Admissions Office, The Open University, P.O. Box 48, Milton Keynes, MK7 6AB.

1.1

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A BRIEF HISTORY OF DIFFERENTIAL GEOMETRY

This is a very brief history of those aspects of differential geometry that are dealt with in this course. It is here to give some historical perspective in a course which developes the modern approach to old questions. Differential geometry is a vast subject and most of the great mathematicians of the last three centuries have made some contribution to it.

Some of the basic problems of differential geometry were already being studied before the introduction of the calculus. For instance, in connection with his study of light, Huygens (1629-95) studied the curvature of certain plane curves. However, the calculus enabled Newton (1647-1727) to make a more detailed study of such curves and Euler (1707-83) was able to study curves on more general surfaces. The study of surfaces themselves was opened up by the use of patches introduced in a restricted way by Monge (1746-1818) and used by Gauss (1777-1855) to describe the curvature of a surface in an intrinsic way. This enabled one to answer such questions as whether it was possible to make a plane map of the globe that retained all the local properties of its geometry.

It was Riemann (1826-66) who saw how the theory of surfaces could be generalized. The study of Riemann surfaces showed finally that Euclidean geometry was not the only possibility. There is now a vast literature on such geometries.

Meanwhile, the theory of curves took a great step forward with the discovery, by Frenet (1816-1900) and Serret (1819-85), of formulas describing the curvature and torsion of curves. Their "method of moving frames" was adapted by Darboux (1842-1907) to describe surfaces. This method was brought to full generality by Cartan (1869-1951), who brought back geometric insight into the profusion of equations thrown up by Riemann and his followers.

A readable introduction to some of the geometric ideas dealt with in this course can be found in:

D. Hilbert and S. Cohn-Vossent: *Geometry and the Imagination* (Chelsea, 1952).

This section on differential geometry contains pictures of several of the surfaces we shall be studying.

BIBLIOGRAPHY AND CONVENTIONS

Bibliography

The set book for this course is Barrett O'Neill: *Elementary Differential Geometry* (Academic Press, 1966). It is essential to have this book: the course is based on it and will not make sense without it.

The set books for M201, M231 and MST 282 are referred to occasionally; they are useful but not essential. They are:

D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins: *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

E.D. Nering: *Linear Algebra and Matrix Theory* (John Wiley, 1970).

M. Spivak: *Calculus,* paperback edition (W.A. Benjamin/Addision-Wesley, 1973).

R.C. Smith and P. Smith: *Mechanics*, SI Edition (John Wiley, 1972).

Conventions

Unreferenced pages and sections denote the set book. Otherwise

O'Neill denotes the set book;

Text denotes the correspondence text;

KKOP denotes *An Introduction to Linear Analysis* by D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins;

Nering denotes *Linear Algebra and Matrix Theory* by E.D. Nering;

Spivak denotes *Calculus* by M. Spivak;

Smith denotes *Mechanics* by R.C. Smith and P. Smith.

Reference to Open University Courses in Mathematics take the form:

Unit M100 22, Linear Algebra I

Unit MST 281 10, Taylor Approximation

Unit M201 16, Euclidean Spaces I: Inner Products

Unit M231 2, Functions and Graphs

Unit MST 282 1, Some Basic Tools.

References to Definitions, Theorems, Lemmas etc. in the set book, O'Neill, are given as in the following example:

Lemma V.4.2 stands for Lemma 4.2 of Chapter V.

Sometimes if O'Neill is referring to something in the same chapter he omits the roman numeral for the Chapter.

PREREQUISITES

The courses that are prerequisites for this course are M201 Linear Mathematics, M231 Analysis, and MST 282 Mechanics and Applied Calculus.

However, this course has been written with very few explicit references to any of these courses. In this section we list those units and sections of the prerequisite courses that we have built on, and this is followed by a brief description of their content. This should enable you to work out whether you are familiar with those ideas and results that we shall be assuming. The bracketed sections below are marked as optional to M231 and so you may not have read them yet. If you want to, do so now, noting only the ideas and results. The details of the definitions and proofs will not be needed and our notation will not be similar.

Prerequisite Sections

M201 Linear Mathematics

Unit 1	Vector Spaces
Unit 2	Linear Transformations
Unit 3	Hermite Normal Form, Section 3.3
Unit 5	Determinants and Eigenvalues
Unit 10	Jordan Normal Form, Section 10.2.1
Unit 12	Linear Functionals and Duality, Section 12.1
Unit 14	Bilinear and Quadratic Forms, Sections 14.1 and 14.2.3
Unit 15	Affine Geometry and Convex Cones, Sections 15.1.1 and 15.1.2
Unit 16	Euclidean Spaces I, Inner Proudcts
Unit 23	The Wave Equation, Section 23.1.1
Unit 24	Orthogonal and Symmetric Transformations, Sections 24.1, 24.3.1 and 24.3.2

M231 Analysis

Unit 2	Functions and Graphs (plus Section 2.7)
Unit 5	Differentiation (plus Section 5.9)
(Unit 7	Applications of the Derivative, Section 7.8)
Unit 10	Some Important Functions

MST 282 Mechanics and Applied Calculus

Unit 1	Some Basic Tools, Sections 1.1. to 1.3
Unit 2	Kinematics, Sections 2.1.1 and 2.2
Unit 7	Work and Energy I, Section 7.1.2

Detailed Prerequisites

M201 Linear Mathematics

This is the major prerequisite for our course. Though we do not often quote results from it, we freely use all the basic ideas and conventions that it introduced.

Unit 1 Vector Spaces This unit introduces the vector space of arrows, R^2 and R^3, which are the basis for our course. You should also know about *linear dependence, bases* and *coordinates*, though we shall deal only with concrete examples in R^3. Ideas such as *dimension* and *subspaces* are implicit in our course but you will not need any explicit results. In fact, the ideas of this unit are basic to all linear mathematics.

Unit 2 Linear Transformations This unit again introduces ideas basic to our course. You will need to know about *linear transformations*, the convention for representing them by *matrices, isomorphisms* and *rank,* though we shall be interested in 2- and 3-dimensional examples only. This is another unit with which to be entirely familiar.

Unit 3 Hermite Normal Form We do not use the technique of Hermite normal forms in our course. However, we do talk about the *rank* of 2×2 and 3×3 matrices, which can be read off the Hermite normal form obtained by the elementary operations in Section 3.3.

Unit 5 Determinants and Eigenvalues *Determinants, eigenvalues* and *eigenvectors* play a small but important part in our course. The cross product of two vectors will be defined in terms of 3×3 determinants and you should be familiar with the formula for expanding such a determinant dealt with in KKOP, pages 682-687, where the effect of column operations is also described. In Section 5.1.5 it is shown that row operations behave similarly and it is these that we shall use. You will need to know only what eigenvectors and eigenvalues are and not in general how to calculate them. We shall be using eigenvector bases as in Section 5.2.4.

Unit 10 Jordan Normal Form In Section 10.2.1 it is shown that a linear transformation is represented by a diagonal matrix with respect to an eigenvector basis.

Unit 12 Linear Functionals and Duality It will be useful to know the definition of *linear functionals,* the *dual space* and a *dual basis.* However, do not worry about any technical results. Since we shall be dealing with R^2 and R^3, in which there is a simple inner (scalar, dot) product, we shall be able to describe dual bases etc. in a concrete way.

Unit 14 Bilinear and Quadratic Forms Again you will find it useful to have met *bilinear forms* and *symmetric bilinear forms.* It will give you some more insight into the course, although the only explicit result we shall use is the quadratic Taylor approximation.

Unit 15 Affine Geometry and Convex Cones Though this unit is not in the spirit of our course, Sections 15.1.1 and 15.1.2 give various ways of describing affine lines and planes which may prove useful to you.

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Unit 14	Bilinear and Quadratic Forms, Sections 14.1 and 14.2.3
Unit 15	Affine Geometry and Convex Cones, Sections 15.1.1 and 15.1.2
Unit 16	Euclidean Spaces I, Inner Proudcts
Unit 23	The Wave Equation, Section 23.1.1
Unit 24	Orthogonal and Symmetric Transformations, Sections 24.1, 24.3.1 and 24.3.2

M231 Analysis

Unit 2	Functions and Graphs (plus Section 2.7)
Unit 5	Differentiation (plus Section 5.9)
(Unit 7	Applications of the Derivative, Section 7.8)
Unit 10	Some Important Functions

MST 282 Mechanics and Applied Calculus

Unit 1	Some Basic Tools, Sections 1.1. to 1.3
Unit 2	Kinematics, Sections 2.1.1 and 2.2
Unit 7	Work and Energy I, Section 7.1.2

Detailed Prerequisites

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Unit 16 Euclidean Spaces I: Inner Products Sections 16.1 to 16.3 are very important to our course. They introduce the *inner* (scalar, dot) *product* of vectors, which we use extensively. You will need to be able to use it to define *length, distance, angle* and *orthonormal bases.* We shall frequently use orthonormal bases, orthonormal expansions and the simple form of the inner product in terms of the coordinates with respect to such a basis, all of which are dealt with in Section 16.4.3. Our course goes over all the details again.

Unit 23 The Wave Equation All you need from this unit is the definition of *partial differention* in Section 23.1.1, though even this will be redefined.

Unit 24 Orthogonal and Symmetric Transformations This unit contains many results that are needed in this course. *Orthogonal transformations* are linear space 'morphisms' and the *orthogonal matrices* representing them are characterized in several important ways, including the result that their inverse is equal to their transpose. We shall also use the result that symmetric transformations can be represented by diagonal matrices with respect to suitable eigenvector bases.

M231 Analysis

Unit 2 Functions and Graphs This unit covers all the basic ideas associated with *graphs* and *functions of one variable* with which you need to be familiar.

The appendix, Section 2.7, is very straightforward and you ought to have a look at it. It introduces ideas that will be developed in this course.

Unit 5 Differentiation This unit introduces the basic definitions of *differentiation* and *curves*. These are ideas which will be developed further in this course. However, our approach will be less theoretical. We shall be interested in using techniques rather than worrying about such problems as differentiability.

We shall also be using *partial* and *directional derivatives* extensively. These are introduced in an appendix, Section 5.9. If you have looked at this before you might like to look at it again, otherwise don't worry.

Unit 7 Applications of the Derivative The only relevant part of this unit is the appendix, Section 7.8, on *implicit functions.* This is straightforward and will help you with Chapter IV of this course. Try to look through it before then but do not bother with any proofs or the sections on contunuity and differentiability.

Unit 10 Some Important Functions This unit introduces all the standard functions that you will need in this course. You will need to know about *polynomials, trigonometric functions,* the *exponential* and *logarthmic functions, hyperbolic functions* and *inverse trigonometric and hyperbolic functions.* Useful relations involving these and their derivatives can be found in the summary, Section 10.10.

MST 282 Mechanics and Applied Calculus

Unit 1 Some Basic Tools Sections 1.1 to 1.3 give a useful introduction to *vectors* and *applied calculus*, with which you should be familiar. We shall use the *scalar, vector* and *triple scalar products* of vectors and some simple forms of the *chain rule* for differentiation.

Unit 2 Kinematics In Section 2.1.1 differentiation of a vector is introduced and in Section 2.2 parametric equations of curves are discussed. These are ideas that will be developed further and so you should be familiar with them.

Unit 7 Work and Energy I Section 7.1.2 contains some very important ideas that we shall develop further and so you should be familiar with them. It deals with *scalar fields, level curves and surfaces, directional derivatives* and the *gradient.* It also contains a simple form of the chain rule for differentiation, for which it gives a proof in Appendix 1.

Handbooks

Each of the prerequisite courses has a handbook which contains some of the definitions to which we refer. However, the most useful parts of the handbooks are those giving lists of the derivatives of standard functions. These are

M231 Handbook: Section 5, Familiar Functions, pages 45-48

MST 282 Handbook: Section 4, Table of Derivatives, page 27.

THE STRUCTURE OF THE COURSE

This course is based on B. O'Neill's book *Elementary Differential Geometry* published by Academic Press. Again we must stress that it is essential to have this book since the course is based almost entirely on it. The course consists of the first six chapters of the book, apart from a few odd sections, and, with one minor exception, you should read the book in the order it is written. We expect you to read it section by section and we have provided a commentary on each relevant section. The commentaries have been bound together chapter by chapter.

There is another way of viewing the structure of the course which might prove useful. All those sections dealing with a particular theme can be thought of as forming a block. We shall show which sets of sections form blocks and also how these blocks are interrelated, that is, which blocks it is essential to have tackled, at least in part, before going onto another block. If you ever become stuck with the sections in one block this will enable you to see if there is any alternative block that you can tackle first.

Sometimes a particular section does not depend on all those sections that have preceded it in that block or on all the sections of preceding blocks. We have included this information in the form of a detailed structure diagram for each block giving all prerequisite blocks and sections.

Each introduction to a section again lists the most important prerequisite sections but we hope that if you refer to the structure diagrams below regularly the overall structure of the course will become clearer.

1. Breakdown into Blocks

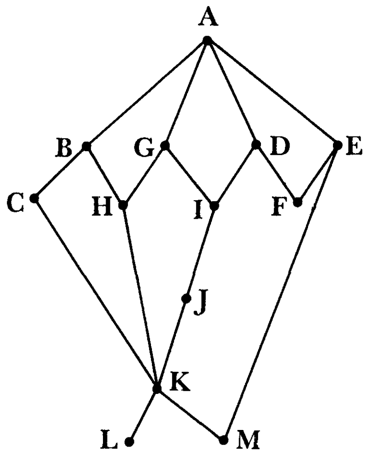
Here is the content of the blocks and the number of units to which they are equivalent, assuming that this is a standard ½ credit course with fourteen teaching units and two gaps.

A.	Calculus and Linear Algebra in E^3 I.1 to I.4, I.7, II.1, II.2, II.5, II.6	3 units
B.	Basic Exterior Calculus in E^3 I.5 and I.6	½ unit
C.	Connection Forms for Frame Fields in E^3 II.7 and II.8	½ unit
D.	Frenet Formulas for Curves in E^3 II.3 and II.4	1 unit
E.	Geometry of E^3 III.1 to III.3	½ unit
F.	Geometry of Curves in E^3 III.4 and III.5	½ unit
G.	Calculus on Surfaces in E^3 IV.1 to IV.3, IV.5(a)	1½ units
H.	Exterior Calculus on Surfaces in E^3 IV.4, IV.5(b), IV.7(a)	1 unit

I.	Shape Operator of a Surface in E^3 V.1 to V.4	2 units
J.	Applications of the Shape Operator V.5 and V.6	1 unit
K.	Structure of Surfaces in E^3 VI.1, VI.2, VI.6	1 unit
L.	Global results on Surfaces in E^3 VI.3, IV.7(b)	½ unit
M.	Geometry of Surfaces in E^3 VI.4, VI.5, VI.8	1 unit

2. Structure of Collection of Blocks

The blocks are related as below, a block is preceded by all those blocks above it to which it is connected.



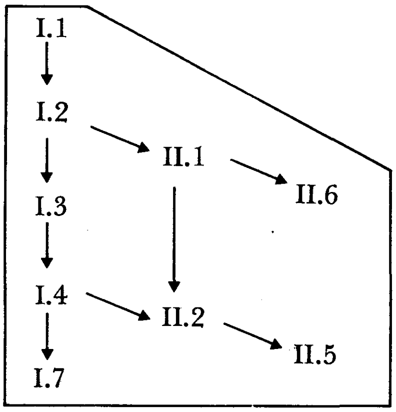
So, for instance, even if you find block **C** too difficult, and on first reading it is quite difficult, there is no reason why you should not press on with **D, E, F, G, H, I, J** as long as you have looked at **C** again before you come to **K** that depends on it.

Each block and all those that precede it tell an interesting story. For instance the line terminating at **J** consists of the blocks **A, D, G, I, J** and describes the shape of surfaces of revolution and interesting curves on surfaces.

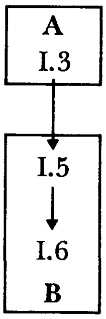
3. Detailed Structure of Blocks and Interconnections

In the following diagrams sections in the same block are grouped together. Arrows indicate how the sections are interconnected. We have shown only how a section depends on those directly preceding it. They in turn may depend on others.

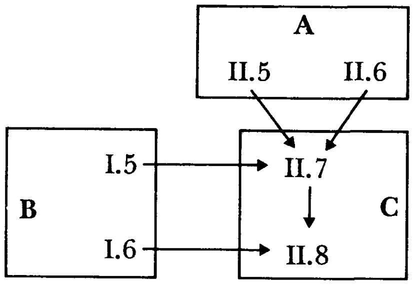
A


I.1
I.2
I.3
I.4
I.7
II.1
II.6
II.2
II.5

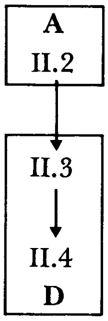
B


A
I.3
I.5
I.6
B

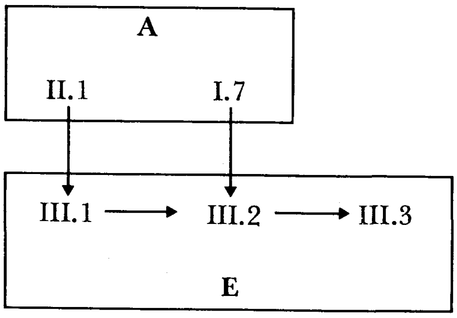
C


A
II.5
II.6
B
I.5
I.6
II.7
II.8
C

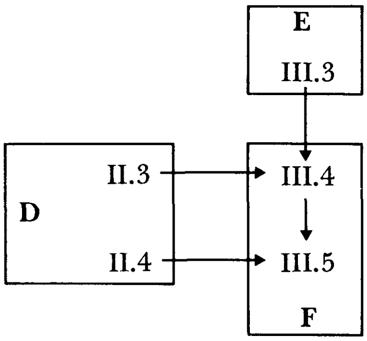
D


A
II.2
II.3
II.4
D

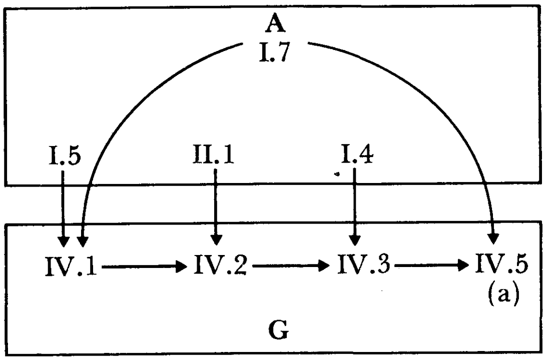
E


A
II.1
I.7
III.1
III.2
III.3
E

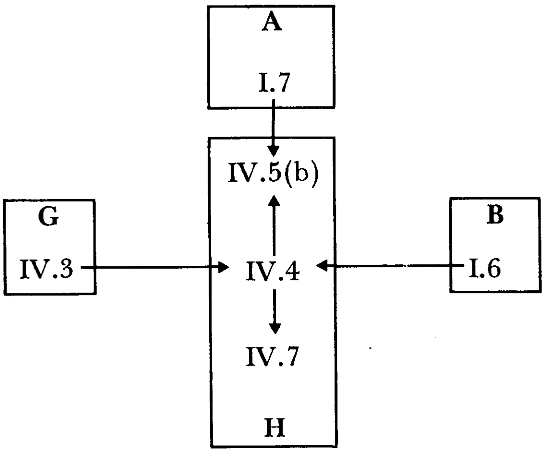
F


E
III.3
D
II.3
II.4
III.4
III.5
F

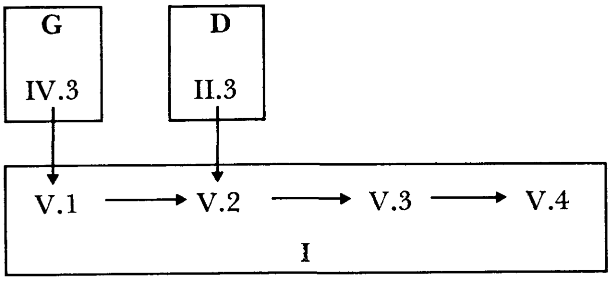
G


A
I.7
I.5
II.1
I.4
IV.1
IV.2
IV.3
IV.5
(a)
G

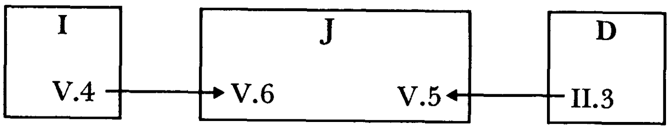
H


A
I.7
IV.5(b)
G
IV.3
IV.4
B
I.6
IV.7
H

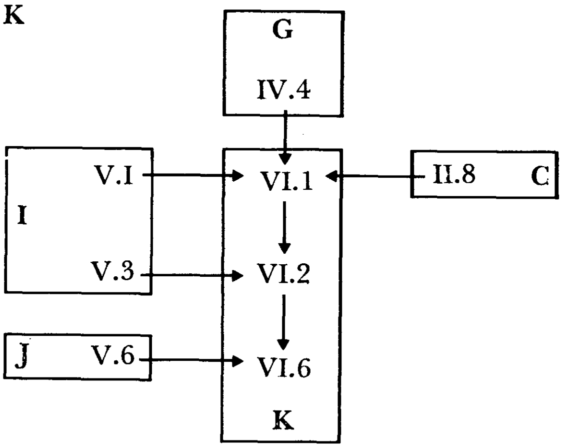
I


G
IV.3
D
II.3
V.1
V.2
V.3
V.4
I

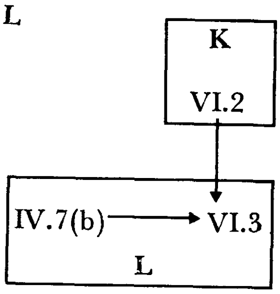
J


I
V.4
J
V.6
V.5
D
II.3

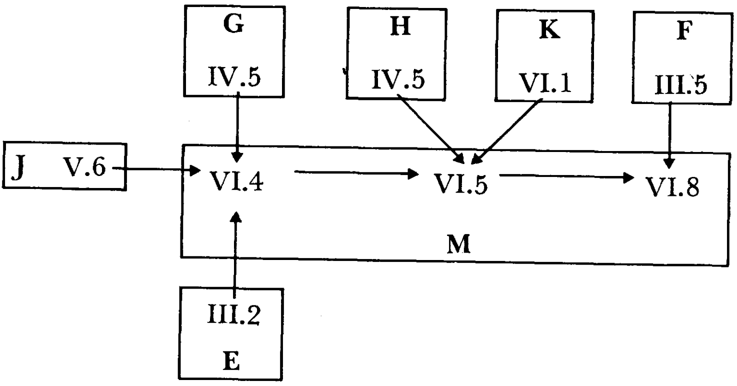
K


G
IV.4
V.I
I
V.3
VI.1
VI.2
VI.6
K
II.8
C
J
V.6

L


K
VI.2
IV.7(b)
VI.3
L

M


G
IV.5
H
IV.5
K
VI.1
F
III.5
J
V.6
VI.4
VI.5
VI.8
M
III.2
E

A POSSIBLE 16-WEEK BREAKDOWN OF THE COURSE

In case you wish to break down the course into 16 weeks of work, the following is a suggested way of doing so.

Week	Sections of O'Neill	Blocks	TMA
1	I.1, I.2, I.3, I.4	**A**	01
2	I.5, I.6, I.7	**A, B**	
3	II.1, II.2	**A**	
4	II.3, II.4	**D**	02
5	II.5, II.6, II.7, II.8	**A, C**	
6			
7	III.1, III.2, III.3, III.4, III.5	**E, F**	
8	IV.1, IV.2	**G**	03
9	IV.3, IV.4	**G, H**	
10	IV.5, IV.7(a)	**G, H**	
11	V.1, V.2, V.3	**I**	
12	V.4	**I**	04
13	V.5, V.6	**J**	
14	VI.1, VI.2, VI.3, IV.7(b)	**K, L**	
15	VI.4, VI.5, VI.6	**K, M**	
16	VI.8	**M**	

WORKING INSTRUCTIONS

This section deals with instructions on how to tackle written material of this course. Of course, we realize that by now that you will probably have worked out which way it is best for you to tackle an Open University text and you may not wish to change successful habits. However, while we were writing our commentaries we had to keep in mind a certain approach to reading them and this was responsible for them being structured in the way they are.

Components

This course has two components. The first is the Set Book, B. O'Neill: *Elementary Differential Geometry*, and the second is the Course Notes. The set book is divided into Chapters which correspond to Parts of the course notes and both of these are divided into Sections indexed in the same way.

Reading a Section

Each section is divided up in the following way.

Introduction

Each section begins with a short introduction. The content of this introduction depends on just how much of an introduction there is in the set book. The introduction tells you the prerequisite sections and possibly some essential results that will be used. It goes on to describe very briefly the content of the section, in terms of those that have gone before, and how this section fits into the overall structure of the course. It will also tell you if it is possible to put this section off till some later time.

Reading Passage

This will usually be an entire section of the set book and it should be read after the introduction. We expect you to read it more than once.

First Reading At the first reading you should try to browse through all the reading passage and obtain an overall view. Look carefully at each definition and try to think of examples. This is a very geometric course so you should try to draw lots of pictures for yourself. Look carefully at any Theorems or Lemmas, making sure you understand all the terms on both sides of each equation, but at this reading do not worry too much about detailed proofs. You should also try and follow the steps in any straightforward worked examples. There are several manipulative techniques that are demonstrated in this way and you will need to master them.

Comments

After completing the reading passage you should next look at the comments we have provided. Though they are related to the text you are not expected to jump backwards and forwards from the reading page to them. In fact some of the comments are quite general and relate to an idea that is developed throughout the section rather than some specific result. You should read these comments in the same spirit as the reading passage.

Additional Text

The Additional Text should be read directly after the Reading Passage and the Comments and approached in exactly the same way. The Additional Text consists of definitions and results that we feel should be discussed at this point and it should be treated as an integral part of the section.

Optional Text

Some sections will contain an Optional Text. These consist either of passages of the text book or of the course notes and develop certain straightforward topics that we consider interesting but peripheral to the main structure of the course. We have included these Optional Texts for those of you who are finding the course enticing and have time to spare; others may omit them without losing anything.

Text Exercises

At the end of the Comments, Additional or Optional Texts you may find some Text Exercises. These are included at this point because they form an integral part of the text. They may deal with some result that has already been referred to or with some result that will be of crucial importance later on.

Second Reading After having worked once through the Reading Passage, Comments and Additional Text you should work through them again but this time in more detail. Once you have grasped the overall structure of the section you need to fill in the finer details. Look up references needed to complete a proof and work through all proofs and examples filling in the missing steps.

Supplementary Comments

It is not essential that you read these comments. They are intended only to help you at the second reading. They are intended to cover those tricky or routine points which are not essential to the development of the subject but may nevertheless cause some concern. Anything not covered by either the Comments or the Supplementary Comments has been omitted because we do not consider it of any importance.

Comments Versus Supplementary Comments The main difference between these two types of comments is that Comments are an integral part of the text and should be included at each reading and during revision, while Supplementary Comments need be looked at only once to clear up tricky points and are not essential to revision.

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Comments Versus Supplementary Comments The main difference between these two types of comments is that Comments are an integral part of the text and should be included at each reading and during revision, while Supplementary Comments need be looked at only once to clear up tricky points and are not essential to revision.

Summary

After mastering as much of the Reading Passage, Comments and Additional Text as you can, you should look through the Summary provided in the Course Notes. Of course if you feel like preparing your own summary and comparing it with ours you might find it a worthwile exercise. Our summary has several subsections. First comes Notation, then Definitions, Results, Examples and Techniques.

Notation We have included only representative examples of each notation introduced in this section.

Definitions and Results We have included in our list of Definitions and Results some that O'Neill has mentioned only in passing but which we feel are important enough to be considered on a par with all the others.

Examples We have included only those examples that are dealt with regularly throughout the course. Important results about these examples are recorded in later Results subsections.

Techniques These techniques could be referred to as the aims of the Sections. For some of the techniques you need to be able to recognize some new object. For others you need to be able to perform some straightforward manipulation, while for others you need to be able to apply some abstract result or reproduce an argument similar to one used in the text.

Exercises

After inspecting the Summary you should try the Exercises without looking at the solutions. The Exercises fall into the following categories.

Text Exercises These were mentioned above.

Technique Exercises These are the most important exercises and are the type we shall expect you to have mastered. We have related each exercise to the relevant techniques so that you know exactly how you ought to tackle it.

Theory Exercises There will be only a few of this type of exercise. They derive some abstract result or slightly extend those dealt with in the Section.

Optional Exercises These are based on or sometimes constitute the Optional Text and so are only for those working through the Optional Texts.

Solutions

The solutions follow the exercises. We have tried to give fairly lengthy solutions to the early exercises, pointing out the reasons for all the steps, but as time goes on we omit reasoning that has been used previously.

Further Exercises

At the end of our commentary on each Chapter there will be a selection of Further Exercises that can be used for self-assessment or revision purposes. They will be either Technique Exercises or Other Recommended Exercises. The techniques needed will be those of the relevant section and each of the other Recommended Exercises will be accompanied by a brief description of how the exercise fits into the course. The Further Exercises will be followed by very brief Solutions.

FURTHER READING

If you are interested in learning more about the subject after you have completed the course, then there are the remaining sections for you to read.

Sections IV.6 and VI.7 deal with integration on a surface in E^3; Section IV.8 introduces manifolds, and Chapter VII applies all the techniques and results of the preceding Chapters to the study of Riemannian Geometry.

O'Neill also provides a short bibliography on page 391. For more differential geometry, at about the same level, you should read the book by T. J. Willmore; for a more advanced approach you should read the book by N. J Hicks, and for applications the book by H. Flanders.

There are a large number of other books now available on differential geometry and it is also the subject of many current research papers. We hope that this course serves you as an introduction to this literature.

ERRATA

The following list comprises the significant mathematical errors we have found in the text of O'Neill. It does not include typographical errors or mistakes where the meaning is nonetheless clear. Errors in exercises and their solutions are generally given not here but where the exercises are mentioned in the correspondence text.

Chapter I

Page 25, last line of text: The coefficient of v_3 should be $(p_2^2 + 2)$.

Page 31, Exercise 6: "φ" in the last line should be "ϑ".

Page 37, line 16: "Corollary 7.6" should be "Theorem 7.5".

Chapter II

Page 54, line 9: The second term of α'' should be $\dfrac{d^2\alpha_2}{dt^2}$

Page 80, Exercise 1: "$\nabla_V W$" should be "$\nabla_v W$".

Chapter III

Page 102, line −2: "$q = F(q)$" should be "$q = F(p)$".

Page 108, line −8: This should read

$$\text{“}\left(\sum_k c_{jk}\, a_{ik}\right) = \left(\sum_k a_{ik}\, {}^t c_{kj}\right) = A\,{}^tC.\text{”}.$$

Page 108, lines −5, −4: The right-hand side of these should read

$$\begin{aligned}\text{“} &= \det(A\,{}^tC)\\ &= \det A.\det {}^tC = \det A.\det C\text{”}.\end{aligned}$$

Page 120, line −7: The last two "*F*"s should be removed.

Chapter IV

Page 136, line −5: The final term should be $\sin^2 v$.

Page 148, Lemma 3.8: This should read

"If $M\colon\ g = c$ is a surface in E^3 and if the gradient vector field $\nabla g = \Sigma\, (\partial g/\partial x_i)\, U_i$ (considered only at points of M) is never zero, then it is a non-vanishing normal vector field on the entire surface M."

The first three lines of the proof should be omitted.

Page 157, Exercise 5. The right-hand side of the first equation should be $g'(f(p))v_p[f]$.

Chapter V

Page 204, lines 9 and −2: "$T_p(M)$" should be "$\overline{T}_p(M)$" in each case.

Page 210, line −9: The final "$\mathbf{x}_u$" should be "$\mathbf{x}_v$".

Page 214, line −6: "-1" should be "$-1/b^2$".

Page 218, line −4 and diagram: "$T_p(M)$" should be "$\overline{T}_p(M)$" in each case.

Page 241, line −5: The value of each curvature should be multiplied by −1.

Chapter VI

Page 252, line 18: The matrix should be transposed.

Page 297, line 6 of text: The second surface should be

$$M\colon z = \frac{x^2-y^2}{2}.$$

Page 299, line −2: The right-hand side of the equation should be

"$\overline{E}_3(F(\mathrm{p}))$".

DIFFERENTIAL GEOMETRY

I Calculus on Euclidean Space

II Frame Fields

III Euclidean Geometry

IV Calculus on a Surface

V Shape Operators

VI Geometry of Surfaces in E^3

O.S.

PART I

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Mathematics: A Third Level Course

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PART I CALCULUS ON EUCLIDEAN SPACE

DIFFERENTIAL GEOMETRY

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

DIFFERENTIAL GEOMETRY PART I

O.U. COURSE UNITS COLLECTION
FOR REFERENCE ONLY

CALCULUS ON EUCLIDEAN SPACE

A commentary on Chapter I of O'Neill's
Elementary Differential Geometry
Prepared by the Course Team

THE OPEN UNIVERSITY PRESS

M334 I

Course Team

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	Dr. R.A. Bailey	Course Assistant in Mathematics
With assistance from:		
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	Mr. G.J. Burt	Lecturer in Educational Technology
	Dr. P.M. Clark	Lecturer in Physics
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For general availability of supporting material referred to in this text, please write to the Director of Marketing, The Open University, P.O. Box 81, Milton Keynes, MK7 6AT.

Further information on Open University courses may be obtained from The Admissions Office, The Open University, P.O. Box 48, Milton Keynes, MK7 6AB.

1.1

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Set Book

Barrett O'Neill: *Elementary Differential Geometry* (Academic Press, 1966). It is essential to have this book: the course is based on it and will not make sense without it.

Bibliography

The set books for M201, M231 and MST 282 are referred to occasionally; they are useful but not essential. They are:

D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins: *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

E.D. Nering: *Linear Algebra and Matrix Theory* (John Wiley, 1970).

M. Spivak: *Calculus,* paperback edition (W.A. Benjamin/Addision-Wesley, 1973).

R.C. Smith and P. Smith: *Mechanics,* SI Edition (John Wiley, 1972).

Conventions

Before starting work on this text, please read M334 Part Zero. Consult the Errata List and the Stop Press and make any necessary alterations for this chapter in the set book.

Unreferenced pages and sections denote the set book. Otherwise

O'Neill	denotes the set book;
Text	denotes the correspondence text;
KKOP	denotes *An Introduction to Linear Analysis* by D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins;
Nering	denotes *Linear Algebra and Matrix Theory* by E.D. Nering;
Spivak	denotes *Calculus* by M. Spivak;
Smith	denotes *Mechanics* by R.C. Smith and P. Smith.

Reference to Open University Courses in Mathematics take the form:

Unit M100 22, Linear Algebra I

Unit MST 281 10, Taylor Approximation

Unit M201 16, Euclidean Spaces I: Inner Products

Unit M231 2, Functions and Graphs

Unit MST 282 1, Some Basic Tools.

I.1 INTRODUCTION AND EUCLIDEAN SPACE

Introduction

In this section we introduce some notation that will be used throughout the course. We review *vector spaces* and *partial differentiation.* In the additional text we give some rules for the evaluation of partial derivatives.

READ: Introduction and Section I.1 (pages 1-5).

Comments

(i) *Page 1: line −1* In the correspondence text we shall sometimes write $g \circ f$ for the composite function.

(ii) *Page 4: Definition 1.2* From now on we shall use x, y, z in this way only. We shall not use x as a typical element of the real line, or indeed any set as O'Neill does on pages 1 and 2 of the Introduction.

Additional Text

Partial differentiation Suppose $p = (p_1, p_2, p_3)$ is a point of E^3 and $f : E^3 \longrightarrow R$ is a differentiable function. To obtain $(\partial f/\partial x_1)(p)$ we first construct the function $F : R \longrightarrow R$ such that $F(t) = f(t, p_2, p_3)$ and then $(\partial f/\partial x_1)(p) = F'(p_1)$. That is we consider x_2 and x_3 as constants and differentiate in the usual way. Partial differentiation satisfies the following rules. Let a, b be real numbers; f, g differentiable functions from E^3 to R and h a differentiable function from R to R, then

(1) $$\frac{\partial(af+bg)}{\partial x_i} = a\frac{\partial f}{\partial x_i} + b\frac{\partial g}{\partial x_i} \qquad (\text{for } i = 1, 2, 3, \ldots)$$

(2) $$\frac{\partial(fg)}{\partial x_i} = \frac{\partial f}{\partial x_i} g + f\frac{\partial g}{\partial x_i} \qquad (\text{for } i = 1, 2, 3, \ldots)$$

(3) $$\frac{\partial(h(f))}{\partial x_i} = h'(f)\frac{\partial f}{\partial x_i} \qquad (\text{for } i = 1, 2, 3, \ldots)$$

Equation (3) may be more recognizable if the composite functions are written as $h \circ f$ and $h' \circ f$. All these results follow from the similar rules for ordinary differentiation. Lists of useful derivatives are to be found in

M231 Handbook, page 45 *et seq.*

MST 282 Handbook, page 27.

Example Find $\dfrac{\partial(2x + xz\sin(x^2y))}{\partial x}$.

We have

$$\frac{\partial(2x + xz\sin(x^2y))}{\partial x} = 2\frac{\partial x}{\partial x} + \frac{\partial(xz\sin(x^2y))}{\partial x}, \qquad \text{by result (1)}$$

$$= 2 + \frac{\partial(xz)}{\partial x}\sin(x^2y) + xz\frac{\partial(\sin(x^2y))}{\partial x}, \qquad \text{by result (2),}$$

$$= 2 + z\sin(x^2y) + xz\sin'(x^2y)\frac{\partial(x^2y)}{\partial x}, \qquad \text{by result (3)}$$

$$= 2 + z\sin(x^2y) + 2x^2yz\cos(x^2y).$$

Summary

Notation

$g(f)$	Page 1, line – 4
$g \circ f$	Text, page 5
f^{-1}	Page 2, line 16
E^3	Page 3, Definition 1.1
p	Page 3, Definition 1.1
(p_1, p_2, p_3)	Page 3, Definition 1.1
$p + q$	Page 3, line – 3
$(p_1, p_2, p_3) + (q_1, q_2, q_3)$	Page 3, line – 3
ap	Page 3, line – 1
$a(p_1, p_2, p_3)$	Page 3, line – 1
x, y, z or x_1, x_2, x_3	Page 4, Definition 1.2
$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$	Page 4, line 24
$f + g$	Page 4, line – 5
fg	Page 4, line – 5
E^2	Page 5, line 22
$E^1 = R$	Page 5, line 23

Definitions

(i)	Euclidean 3-space E^3	Page 3, Definition 1.1
(ii)	Sum of points of E^3	Page 3, line – 5
(iii)	Scalar product of a point of E^3	Page 3, line – 2
(iv)	Euclidean plane E^2	Page 5, line 22
(v)	Real line $E^1 = R$	Page 5, line 23
(vi)	Natural coordinate functions x, y, z or x_1, x_2, x_3	Page 4, Definition 1.2
(vii)	Pointwise addition and multiplication of functions	Page 4, line – 5
(viii)	Composite of functions $g(f)$	Page 1, line – 4
(ix)	Differentiable function	Page 4, Definition 1.3
(x)	Inverse function f^{-1}	Page 2, line 16

Results

Partial differentiation satisfies the following rules.

$$(1) \quad \frac{\partial(af+bg)}{\partial x_i} = a\frac{\partial f}{\partial x_i} + b\frac{\partial g}{\partial x_i} \qquad (i = 1, 2, 3, \ldots)$$

$$(2) \quad \frac{\partial(fg)}{\partial x_i} = \frac{\partial f}{\partial x_i}g + f\frac{\partial g}{\partial x_i} \qquad (i = 1, 2, 3, \ldots)$$

$$(3) \quad \frac{\partial(h(f))}{\partial x_i} = h'(f)\frac{\partial f}{\partial x_i} \qquad (i = 1, 2, 3, \ldots)$$

where a, b are real numbers, $f, g : E^3 \longrightarrow R$ and $h : R \longrightarrow R$. Text: page 5

Techniques

(i) Evaluation of a function.
(ii) Simplification of a function.
(iii) Partial differentiation, using results (1), (2), (3).

Exercises

Technique (i)

1. Page 5, Exercises 2(b) and 2(c).

Technique (ii)

2. Page 5, Exercise 1(a).
3. Page 6, Exercise 4(a). Just simplify this function.

Technique (iii)

4. Page 5, Exercises 1(c) and 1(d).
5. Page 6, Exercises 3(a) and 3(b).
6. Page 6, Exercise 4(a).

Solutions

1. Page 5, Exercises 2(b) and 2(c).

If p is a point of E^3, then by the pointwise definition of addition and multiplication of functions

$$f(p) = ((x(p))^2\ y(p)) - ((y(p))^2\ z(p)).$$

By Definition 1.2

$$x((3,-1,\tfrac{1}{2})) = 3, \qquad y((3,-1,\tfrac{1}{2})) = -1 \quad \text{and} \quad z((3,-1,\tfrac{1}{2})) = \tfrac{1}{2}.$$

Hence

$$f((3,-1,\tfrac{1}{2})) = (3^2.(-1)) - ((-1)^2.\tfrac{1}{2}) = -9\tfrac{1}{2}.$$

As in the above result

$$f((a, 1, 1-a)) = a^2.1 - 1^2.(1-a) = a^2 + a - 1, \text{ for any real number } a.$$

2. Page 5, Exercise 1(a).

If $f = x^2 y$ then, since f is defined by pointwise multiplication, we have

$$f(p) = x^2(p)\ y(p) = p_1^2\, p_2, \text{ where } p = (p_1, p_2, p_3) \text{ is a point of } E^3.$$

The function $g = y \sin z$ is defined in terms of composition and pointwise multiplication and so

$$g(p) = y(p) \sin (z(p)) = p_2 \sin p_3.$$

Hence

$$\begin{aligned}(fg^2)(p) &= f(p)(g(p))^2\\ &= p_1^2 p_2\ (p_2 \sin p_3)^2\\ &= p_1^2 p_2^3 \sin^2 p_3\end{aligned}$$

and

$$fg^2 = x^2y^3 \sin^2 z.$$

In this way we see that pointwise addition and multiplication behave exactly as expected.
If $f = x^2y$ and $g = y \sin z$ then

$$fg^2 = x^2y(y \sin z)^2 = x^2y^3\sin^2 z.$$

3. Page 6, Exercise 4(a).
If $h = x^2 - yz$ then, by the pointwise definitions of addition and multiplication of functions

$$h(p) = p_1^2 - p_2p_3, \text{ for any point } p = (p_1, p_2, p_3) \text{ in } E^3.$$

Since $f(p) = h(g_1(p), g_2(p), g_3(p))$ it follows that

$$f(p) = (g_1(p))^2 - (g_2(p))(g_3(p)).$$

Hence $f = g_1^2 - g_2g_3$.

In this exercise $g_1 = x + y$, $g_2 = y^2$, $g_3 = x + z$ and so

$$\begin{aligned} f &= (x + y)^2 - y^2(x + z) \\ &= x^2 + 2xy + y^2 - xy^2 - y^2z. \end{aligned}$$

4. Page 5, Exercises 1(c) and 1(d).
Now $fg = x^2y^2 \sin z$.
Treating x and y as constants we find that

$$\frac{\partial(fg)}{\partial z} = x^2y^2 \cos z.$$

Treating x and z as constants we find that

$$\frac{\partial^2(fg)}{\partial y\partial z} = 2x^2y \cos z.$$

Now $\sin f = \sin x^2y$.
Treating x as a constant we find that

$$\frac{\partial(\sin f)}{\partial y} = x^2 \cos x^2y.$$

5. Page 6, Exercises 3(a) and 3(b).
If $f = x \sin(xy) + y \cos(xz)$ then, treating y and z as constants,

$$\frac{\partial f}{\partial x} = xy \cos(xy) - yz \sin(xz).$$

To deal with Exercise 3(b) we need to extend result (3) to cover the partial differentiation of the composite of three functions, h from E^3 to R and h_1, h_2 from R to itself. In this exercise it will be more convenient to use the symbol $\circ$ for the composite of functions.

$$\frac{\partial(h_2 \circ (h_1 \circ h))}{\partial x} = (h_2' \circ (h_1 \circ h)) \frac{\partial(h_1 \circ h)}{\partial x}, \qquad \text{by result (3)},$$

$$= (h_2' \circ (h_1 \circ h)). (h_1' \circ h)\frac{\partial h}{\partial x}, \qquad \text{by result (3)}.$$

Here $h = x^2 + y^2 + z^2$, $h_1 = \exp$, $h_2 = \sin$

and $f = \sin \circ (\exp \circ (x^2 + y^2 + z^2)) = h_2 \circ (h_1 \circ h)$. Hence

$$\frac{\partial f}{\partial x} = (\sin' \circ (\exp \circ (x^2 + y^2 + z^2)). \exp' \circ (x^2 + y^2 + z^2). \frac{\partial(x^2 + y^2 + z^2)}{\partial x}$$

$$= \cos(e^{(x^2+y^2+z^2)})\ e^{(x^2+y^2+z^2)}2x$$

$$= 2x\, e^{(x^2+y^2+z^2)} \cos(e^{(x^2+y^2+z^2)}).$$

6. Page 6, Exercise 4(a).
From Exercise 3

$$f = x^2 + 2xy + y^2 - xy^2 - y^2 z$$

hence

$$\frac{\partial f}{\partial x} = 2x + 2y - y^2.$$

I.2 TANGENT VECTORS

Introduction

In this section we review *arrows*. They will be called *tangent vectors*. At each point of Euclidean space they form a vector space. We also review *vector fields*. They are functions assigning to each point of Euclidean space a tangent vector based at that point. We generalize the notion of basis and coordinates to deal with vector fields.

READ: Section I.2 (pages 6–10).

Comments

(i) *Page 9: Definition 2.4* Whereas a vector field V assigns *one* tangent vector to each point of Euclidean space, the natural frame field assigns *three.*

(ii) *Page 10: end of Section* Since any vector field V can be expressed as $V = \Sigma v_i U_i$, for suitable real valued functions v_1, v_2 and v_3 the natural frame field does serve as a 'basis'. When we used the term *basis* for a real vector space it implied that every vector could be expressed as a linear combination of the basis vectors. The values of the vector fields U_1, U_2, U_3 are linearly independent at each point. However, not every vector field can be expressed as a linear combination of the U_i's with constant coefficients. That is the coefficients in the sum $\Sigma v_i U_i$ are not necessarily real numbers. Now they are real valued functions and the collection of real valued functions is not even a field since it is not possible to find a multiplicative inverse of a function which is zero at some points even if it is not always zero. These functions form a *ring* and the vector fields are a finite-dimensional *module* over this ring.

Supplementary Comment

(i) *Page 10: lines 3 and 4* These are generalizations of the usual rules for the addition and scalar multiplication of vectors expressed in terms of coordinates.

The first equation is obtained as follows:

$$(\Sigma(v_i + w_i)U_i)(p) = \Sigma(v_i + w_i)(p)U_i(p),$$

by repeated use of the pointwise formulas for addition and scalar multiplication of vector fields,

$$= \Sigma(v_i(p) + w_i(p))U_i(p),$$

by the definition of pointwise addition of two functions,

$$= \Sigma v_i(p)U_i(p) + \Sigma w_i(p)U_i(p),$$

by the usual rules for a vector space,

$$= (\Sigma v_i U_i)(p) + (\Sigma w_i U_i)(p)$$

$$= (\Sigma v_i U_i + \Sigma w_i U_i)(p),$$

by the pointwise rules for the "arithmetic" of vector fields.

Since vector fields are defined as functions on E^3, whenever two vector fields agree at each point they must be the same vector field.

Hence

$$\Sigma(v_i + w_i)U_i = \Sigma v_iU_i + w_iU_i.$$

The result

$$\Sigma(fv_i)U_i = f(\Sigma v_iU_i)$$

follows similarly.

Summary

Notation

v_p	Page 6, Definition 2.1
$T_p(E^3)$	Page 7, Definition 2.2
$v_p + w_p$	Page 7, line 9
av_p	Page 7, line 9
V	Page 8, Definition 2.3
V + W	Page 8, line -8
fV	Page 8, line -3
$\{U_1, U_2, U_3\}$	Page 9, Definition 2.4
Σv_iU_i	Page 9, line -8

Definitions

(i)	Tangent vector v_p	Page 6, Definition 2.1
(ii)	Vector part of tangent vector	Page 6, Definition 2.1
(iii)	Point of application of tangent vector	Page 6, Definition 2.1
(iv)	Parallel tangent vectors	Page 6, line -5
(v)	Tangent space $T_p(E^3)$	Page 7, Definition 2.2
(vi)	Addition and scalar multiplication of tangent vectors	Page 7, line 9
(vii)	Vector field V	Page 8, Definition 2.3
(viii)	Pointwise principle	Page 8, line -7
(ix)	Addition of vector fields	Page 8, line -10
(x)	Multiplication of a vector field by a real-valued function	Page 8, line -3
(xi)	Natural frame field $\{U_1, U_2, U_3\}$	Page 9, Definition 2.4
(xii)	Euclidean coordinate functions	Page 9, Lemma 2.5
(xiii)	Differentiable vector field	Page 10, line 6

Results

(i)	$T_p(E^3)$ is linearly isomorphic to E^3.	Page 7, line −3
(ii)	Each vector field V can be uniquely expressed as $V = \Sigma v_i U_i$, where v_1, v_2, v_3 are real-valued functions.	Page 9, Lemma 2.5
(iii)	Each tangent vector $(a_1, a_2, a_3)_p$ can be expressed in terms of the natural frame field as $\Sigma a_i U_i(p)$.	Page 9, line −5
(iv)	Addition and multiplication by a function are expressed in terms of coordinates by $\Sigma v_i U_i + \Sigma w_i U_i = \Sigma(v_i + w_i)U_i$ $f(\Sigma v_i U_i) = \Sigma(f v_i)U_i$.	Page 10, lines 3 and 4

Techniques

(i)	Representation of tangent vectors in standard form v_p.	Page 6, Definition 2.1
(ii)	Addition and scalar multiplication of tangent vectors.	Page 7, line 9
(iii)	Evaluation of a vector field at a point.	Page 8, Definition 2.3
(iv)	Addition and multiplication, by real-valued functions, of vector fields.	Page 8, line -8 and Page 9, line 2
(v)	Pictorial representation of tangent vectors and vector fields.	Page 7, Fig. 1.1
(vi)	Representation of tangent vectors and vector fields in terms of the natural frame field and Euclidean coordinates.	Page 9, Lemma 2.5
(vii)	Addition and multiplication, by a real-valued function, of vector fields expressed in terms of Euclidean coordinates.	Page 10, lines 3 and 4

Exercises

Techniques (ii), (vi) and (v)

1. Page 10, Exercise 1.

Techniques (iii) and (vii)

2. Page 10, Exercise 2.

Techniques (vii), (vi), (iv) and (i)

3. Page 10, Exercise 3.

Solutions

1. Page 10, Exercise 1.

(a)

$$\begin{aligned} 3\mathbf{v} - 2\mathbf{w} &= 3(-2, 1, -1) - 2(0, 1, 3) \\ &= (-6, 3, -3) - (0, 2, 6) \\ &= (-6, 1, -9). \end{aligned}$$

By the definition of addition and scalar multiplication of tangent vectors,

$$3\mathbf{v}_\mathbf{p} - 2\mathbf{w}_\mathbf{p} = (3\mathbf{v} - 2\mathbf{w})_\mathbf{p} = (-6, 1, -9)_\mathbf{p}.$$

Using the identity $(a_1, a_2, a_3)_\mathbf{p} = \Sigma a_i U_i(\mathbf{p})$, we find that

$$3\mathbf{v}_\mathbf{p} - 2\mathbf{w}_\mathbf{p} = -6U_1(\mathbf{p}) + U_2(\mathbf{p}) - 9U_3(\mathbf{p}).$$

(b) $\mathbf{v}_\mathbf{p}$ represents the arrow from the point $\mathbf{p}$ to the point $\mathbf{p} + \mathbf{v}$. Here $\mathbf{p} = (1, 1, 0)$, $\mathbf{v} = (-2, 1, -1)$ and hence $\mathbf{p} + \mathbf{v} = (-1, 2, -1)$. Similarly, $\mathbf{w}_\mathbf{p}$ is the arrow from $\mathbf{p} = (1, 1, 0)$ to $\mathbf{p} + \mathbf{w} = (1, 2, 3)$.

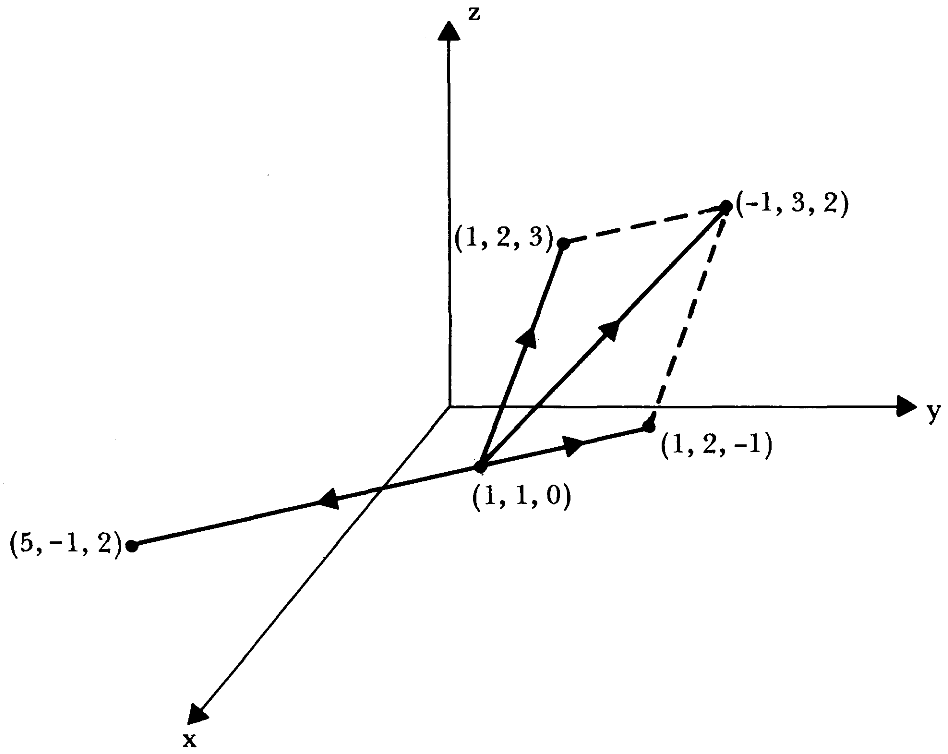
Now

$$-2\mathbf{v}_\mathbf{p} = (-2\mathbf{v})_\mathbf{p} = (4, -2, 2)_\mathbf{p}$$

and

$$\mathbf{v}_\mathbf{p} + \mathbf{w}_\mathbf{p} = (\mathbf{v} + \mathbf{w})_\mathbf{p} = (-2, 2, 2)_\mathbf{p}$$

and hence they are the arrows from (1, 1, 0) to (5, -1, 2) and (-1, 3, 2).



2. Page 10, Exercise 2.

$$\begin{aligned} W - xV &= 2x^2U_2 - U_3 - x(xU_1 + yU_2) \\ &= 2x^2U_2 - U_3 - x^2U_1 - xyU_2 \\ &= -x^2U_1 + (2x^2 - xy)U_2 - U_3, \end{aligned}$$

by result (iv) on the addition and multiplication of vector fields in terms of coordinates.

If $p = (-1, 0, 2)$ then $x(p) = -1$, $y(p) = 0$ and $z(p) = 2$.

By the definition of the natural frame field

$U_1(p) = (1, 0, 0)_p$, $U_2(p) = (0, 1, 0)_p$, $U_3(p) = (0, 0, 1)_p$. Hence by the pointwise definition of addition and multiplication

$$\begin{aligned} (W - xV)((-1, 0, 2)) &= -(-1)^2(1, 0, 0)_p + (2.(-1)^2 - (-1).0)(0, 1, 0)_p - (0, 0, 1)_p \\ &= (-1, 2, -1)_p. \end{aligned}$$

3. Page 10, Exercise 3.

(a) Rearranging and dividing by 7 we obtain

$$V = \frac{2z^2}{7}U_1 - \frac{xy}{7}U_3.$$

(b) $V(p) = (p_1, p_3 - p_1, 0)_p$

$= (x_1(p), x_3(p) - x_1(p), 0)_p,$ using the definition of the coordinate functions,

$= (x_1(p), (x_3 - x_1)(p), 0)_p,$ by the definition of linear combinations of functions,

$= x_1(p)U_1(p) + (x_3 - x_1)(p)U_2(p),$ by the standard identity $(a_1, a_2, a_3)_p = \Sigma a_iU_i(p)$.

Hence, using the definition of linear combinations of vector fields,

$$V = x_1U_1 + (x_3 - x_1)U_2.$$

Alternatively we could write this as

$$V = xU_1 + (z - x)U_2.$$

(c) $V = 2(xU_1 + yU_2) - x(U_1 - y^2U_3)$

$= 2xU_1 + 2yU_2 - xU_1 + xy^2U_3,$ by result (iii) on addition and multiplication in terms of coordinates,

$= (2x - x)U_1 + 2yU_2 + xy^2U_3$

$= xU_1 + 2yU_2 + xy^2U_3.$

(d) If $p = (p_1, p_2, p_3)$ then $V(p)$ is the vector from (p_1, p_2, p_3) to $(1 + p_1, p_2p_3, p_2)$.

Hence $V(\mathbf{p}) = ((1 + p_1, p_2p_3, p_2) - (p_1, p_2, p_3))_{\mathbf{p}}$

$= (1, p_2p_3 - p_2, p_2 - p_3)_{\mathbf{p}}$.

By arguments similar to those in part (b) we find that

$$V = U_1 + x_2(x_3 - 1)U_2 + (x_2 - x_3)U_3$$

or alternatively

$$V = U_1 + y(z - 1)U_2 + (y - z)U_3.$$

(e) The vector from $\mathbf{p}$ to the origin is $\mathbf{0} - \mathbf{p} = -\mathbf{p}$ so

$$V(\mathbf{p}) = (-\mathbf{p})_{\mathbf{p}} = (-p_1, -p_2, -p_3)_{\mathbf{p}}.$$

Hence $V = - x_1U_1 - x_2U_2 - x_3U_3$ or alternatively

$= - xU_1 - yU_2 - zU_3$.

I.3 DIRECTIONAL DERIVATIVES

Introduction

This section follows on from Sections I.1 and I.2. In Section I.1 we reviewed partial differentiation. We recalled that for any differentiable function f on E^3 and for any point p in E^3

$$\frac{\partial f}{\partial x_1}(p) = \frac{d}{dt}(f(t, p_2, p_3))\Big|_{t = p_1}.$$

Instead of evaluating this derivative at $t = p_1$ we often 'shift' the function so that we can evaluate the derivative at t = 0. That is

$$\frac{\partial f}{\partial x_1}(p) = \frac{d}{dt}(f(p_1 + t, p_2, p_3))\Big|_{t = 0}.$$

Using the natural frame field introduced in Section I.2 we can write this as

$$\frac{\partial f}{\partial x_1}(p) = \frac{d}{dt}(f(p + tU_1(p)))\Big|_{t = 0}.$$

There are similar formulas for $(\partial f/\partial x_2)(p)$ and $(\partial f/\partial x_3)(p)$, so to find $(\partial f/\partial x_i)(p)$ we find the initial rate of change of the function f in the direction $U_i(p)$. In *Unit MST 282 7, Work and Energy I,* Section 7.1.2 we met a generalization of this technique. If **n** was a unit direction vector we defined the directional derivative, $(\partial f/\partial \mathbf{n})(p)$, to be the initial rate of change of f in the direction **n** and proved that

$$\frac{\partial f}{\partial \mathbf{n}} = \mathbf{n} \cdot \nabla f(p) = \sum_{i=1}^{3} n_i \frac{\partial f}{\partial x_i}(p).$$

In this section we generalize the above definition and result to deal with any tangent vector $\mathbf{v}_p$ and then any vector field V. We show that these directional derivatives satisfy properties similar to those of ordinary derivatives.

READ: Section I.3 (pages 11-14) omitting the proof of Lemma 3.2.

Comments

(i) *Page 12: Lemma 3.2* This is a generalization of the result mentioned in the introduction. The directional derivative is

$$\mathbf{v}_p[f] = \mathbf{v} \cdot \nabla f(p) = \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i}(p).$$

It is proved in exactly the same way using a version of the 'chain rule'. This will be dealt with in Section I.7 and until then you should accept this result without proof.

(ii) *Page 12: Theorem 3.3.(3)* The symbol $\cdot$ in this theorem and in Corollary 3.4 signifies the ordinary product of two numbers. The inner product of two vectors will not be introduced until Chapter II.

(iii) *Page 12: line 15 to line -3* The real-valued function V[f] is defined by

$$(V[f])(p) = V(p)[f].$$

This equation is used in the proof of Corollary 3.4.

For computational purposes the identities $U_i[f] = \partial f/\partial x_i$ $(i = 1, 2, 3)$ will be very important. We can obtain these results from Lemma 3.2. For instance

$$U_1(p)\,[f] = (1, 0, 0)_p[f] = 1.\frac{\partial f}{\partial x_1}(p) + 0.\frac{\partial f}{\partial x_2}(p) + 0.\frac{\partial f}{\partial x_3}(p) = \frac{\partial f}{\partial x_1}(p).$$

(iv) *Page 14; line 14 to line 21* This is a very important remark. For vector fields V and W we consider combinations of the form fV + gW, where f and g are real-valued functions. For functions f and g we consider combinations of the form af + bg, where a and b are real numbers.

In the example

$$(xU_1 - y^2U_3)[x^2y + z^3] = xU_1[x^2y + z^3] - y^2U_3[x^2y + z^3], \qquad \text{by Corollary 3.4.(1)},$$

$$= xU_1[x^2y] + xU_1[z^3] - y^2U_3[x^2y] - y^2U_3[z^3], \qquad \text{by Corollary 3.4.(2)}.$$

Supplementary Comment

(i) *Page 12: Theorem 3.3.(1) and (2)*

(1) If $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ then

$$(av_p + bw_p)[f] = (av + bw)_p[f], \qquad \text{by the definition of addition and scalar multiplication of tangent vectors},$$

$$= \Sigma(av_i + bw_i)\frac{\partial f}{\partial x_i}(p), \qquad \text{by Lemma I.3.2},$$

$$= a\left(\Sigma v_i\frac{\partial f}{\partial x_i}(p)\right) + b\left(\Sigma w_i\frac{\partial f}{\partial x_i}(p)\right)$$

$$= av_p[f] + bw_p[f].$$

(2)
$$v_p[af + bg] = \Sigma v_i\left(\frac{\partial(af + bg)}{\partial x_i}\right)(p), \qquad \text{by Lemma I.3.2},$$

$$= \Sigma v_i\left(a\frac{\partial f}{\partial x_i}(p) + b\frac{\partial g}{\partial x_i}(p)\right)$$

$$= a\left(\Sigma v_i\frac{\partial f}{\partial x_i}(p)\right) + b\left(\Sigma v_i\frac{\partial g}{\partial x_i}(p)\right).$$

Summary

Notation

$v_p[f]$	Page 11, Definition 3.1
$V[f]$	Page 13, line 15

Definitions

(i)	Derivative of a function with respect to a tangent vector $v_p[f]$.	Page 11, Definition 3.1
(ii)	Derivative of a function with respect to a vector field $V[f]$.	Page 13, line 15

Results

(i)	If $v_p = (v_1, v_2, v_3)_p$ then $v_p[f] = \Sigma v_i \frac{\partial f}{\partial x_i}(p)$.		Page 12, Lemma 3.2
(ii)	Directional derivatives, with respect to tangent vectors and vector fields, satisfy linear and Leibnizian properties:		
	(a)	$(av_p + bw_p)[f] = av_p[f] + bw_p[f]$.	
	(b)	$v_p[af + bg] = av_p[f] + bv_p[g]$.	
	(c)	$v_p[fg] = v_p[f]\,g(p) + f(p)v_p[g]$.	Page 12, Theorem 3.3
	(a)	$(fV + gW)[h] = fV[h] + gW[h]$.	
	(b)	$V[af + bg] = aV[f] + bV[g]$.	
	(c)	$V[fg] = V[f].g + fV[g]$.	Page 13, Corollary 3.4.
(iii)	Directional derivatives with respect to components of the natural frame field are partial derivatives $U_i[f] = \frac{\partial f}{\partial x_i}$ (i = 1, 2, 3).		Page 13, line -10

Techniques

Evaluation of directional derivatives with respect to tangent vectors or vector fields using:

(i)	the definition, for tangent vectors	Page 11, Definition 3.1
(ii)	the definition, for vector fields	Text, page 16
(iii)	coordinates and partial derivatives	Page 12, Lemma 3.2
(iv)	the linear and Leibnizian properties and the relationship between the natural frame fields and partial derivatives.	Page 12, Theorem 3.3, Page 13, Corollary 3.4 and Page 13, line -10.

Exercises

Technique (i)

1. Page 14, Exercise 1(c).

Technique (ii)

2. Determine $V[f]$ where $f = e^x \cos y$ and $V = 2U_1 - U_2 + 3U_3$.

Technique (iii)

3. Page 15, Exercise 2(c).

Technique (iv)

4. Page 15, Exercise 3(e). First prove that $V[f^2] = 2f\,V[f]$.

Theory Exercises

5. Page 15, Exercise 4.

6. Page 15, Exercise 5. (HINT: Use Exercise 4.)

Solutions

1. Page 14, Exercise 1(c).

   By Definition I.3.1,

$$v_p[f] = \frac{d}{dt}(f(p + tv))\Big|_{t=0}.$$

Here

$$p + tv = (2, 0, -1) + t(2, -1, 3) = (2 + 2t, -t, -1 + 3t)$$

and

$$f(p + tv) = (e^x \cos y)\,(2 + 2t, -t, -1 + 3t)$$

$$= e^{2+2t}\cos(-t),$$ by the pointwise definition of addition, multiplication and composition of functions and the definition of the functions x and y,

$$= e^2 e^{2t}\cos t.$$

Hence

$$v_p[f] = e^2\frac{d}{dt}(e^{2t}\cos t)\Big|_{t=0}$$

$$= e^2(2e^{2t}\cos t - e^{2t}\sin t)\Big|_{t=0} = 2e^2.$$

2. The value of the function $V[f]$ at a point p is given by the formula $(V[f])(p) = V(p)[f]$.

   Now $V(p) = 2U_1(p) - U_2(p) + 3U_3(p) = (2, -1, 3)_p$.

   Hence if $p = (p_1, p_2, p_3)$ then

$$V(p)[f] = \frac{d}{dt}(f((p_1, p_2, p_3) + t(2, -1, 3)))\Big|_{t=0}$$

$$= \frac{d}{dt}(f(p_1 + 2t, p_2 - t, p_3 + 3t))\Big|_{t=0}$$

$$= \frac{d}{dt}(e^{p_1+2t}\cos(p_2 - t))\Big|_{t=0}$$

$$= (2e^{p_1+2t}\cos(p_2 - t) + e^{p_1+2t}\sin(p_2 - t))\Big|_{t=0}$$

$$= e^{p_1}(2\cos p_2 + \sin p_2).$$

So $V[f](p) = e^{p_1}(2\cos p_2 + \sin p_2)$ and hence

$$V[f] = e^x(2\cos y + \sin y).$$

3. Page 15, Exercise 2(c).
By Lemma I.3.2

$$v_p[f] = \Sigma v_i \frac{\partial f}{\partial x_i}(p),$$

though in this solution we replace x_1, x_2, x_3 by x, y, z. Since $f = e^x\cos y$ it follows that

$$\frac{\partial f}{\partial x} = e^x\cos y, \qquad \frac{\partial f}{\partial y} = -e^x\sin y \quad \text{and} \quad \frac{\partial f}{\partial z} = 0.$$

Since $p = (2, 0, -1)$

$$\frac{\partial f}{\partial x}(p) = e^2, \qquad \frac{\partial f}{\partial y}(p) = 0 \quad \text{and} \quad \frac{\partial f}{\partial z}(p) = 0.$$

Hence, since $v = (2, -1, 3)$

$$v_p[f] = 2.e^2 + (-1).0 + 3.0 = 2e^2.$$

4. Page 15, Exercise 3(e).
By the Leibnizian property, Corollary I.3.4(3),

$$V[f^2] = V[f.f] = V[f].f + f\,V[f] = 2f\,V[f].$$

Hence by the linearity property, Corollary I.3.4(2),

$$V[f^2 + g^2] = 2(f\,V[f] + g\,V[g]).$$

Now

$$\begin{aligned} V[f] &= (y^2U_1 - xU_3)[xy] = y^2U_1[xy] - xU_3[xy] \\ &= y^2\frac{\partial(xy)}{\partial x} - x\frac{\partial(xy)}{\partial z}, \qquad \text{since } U_i[f] = \frac{\partial f}{\partial x_i}, \\ &= y^2.y - x.0 = y^3. \end{aligned}$$

Similarly,

$$\begin{aligned} V[g] &= (y^2U_1 - xU_3)[z^3] \\ &= -3xz^2 \end{aligned}$$

and hence

$$\begin{aligned} V[f^2 + g^2] &= 2(xy.(y^3) + z^3.(-3xz^2)) \\ &= 2x(y^4 - 3z^5). \end{aligned}$$

5. Page 15, Exercise 4.
Following the hint we evaluate $V[x_j]$, where $V = \Sigma v_iU_i$.

For fixed j we have

$$V[x_j] = \Sigma v_iU_i[x_j] = \Sigma v_i \frac{\partial x_j}{\partial x_i}.$$

Now $\frac{\partial x_j}{\partial x_i} = 1$ if $i = j$ and is zero otherwise. Hence

$$V[x_j] = v_j \qquad \text{for all } j,$$

and

$$V = \Sigma v_i U_i = \Sigma V[x_i] U_i.$$

6. Page 15, Exercise 5.
Since $V[f] = W[f]$ for every function f on E^3 it follows that

$$V[x_i] = W[x_i] \qquad (i = 1, 2, 3).$$

Hence by the result of Exercise 4

$$V = \Sigma V[x_i] U_i = \Sigma W[x_i] U_i = W.$$

I.4 CURVES IN E^3

Introduction

This section follows on from Sections I.2 and I.3. We review two ways of defining curves, the definition of the velocity of a curve, and show how the rate of change of a function along a curve is related to the directional derivative with respect to the velocity vector.

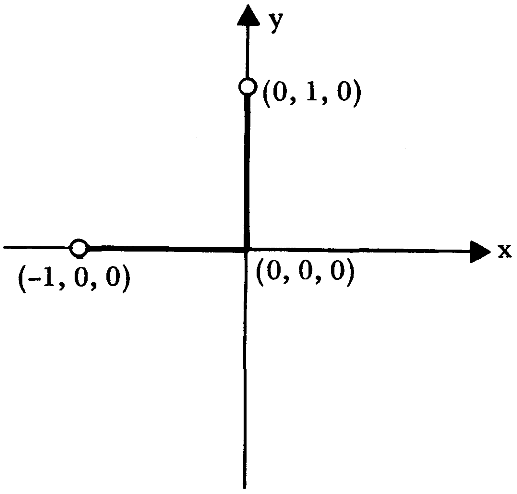
READ: Section I. 4 (pages 15-21) omitting the proof of Lemma 4.6.

Comments

(i) *Section I.4* From Definition 4.1 you might think that curves always have routes that appear in some sense smooth. However, curves can be degenerate in several different ways. For instance, the simplest curve is the constant curve $\alpha : t \longmapsto 0$, which has as its route a single point. The curve $\alpha : t \longmapsto (\sin t, 0, 0)$ has as its route a closed interval which it traverses infinitely often. It is even possible to have a curve that turns sharp corners. For instance, the curve

$$\alpha : t \longmapsto \begin{cases} (-e^{-1/t^2}, 0, 0) & \text{for } t < 0 \\ (0, 0, 0) & \text{for } t = 0 \\ (0, e^{-1/t^2}, 0) & \text{for } t > 0 \end{cases}$$

has the following route



This function was chosen to ensure that the Euclidean coordinate functions are 'infinitely' differentiable. In order for the curve to stand still, reverse or turn a sharp corner, as in the above examples, it is necessary for the velocity vector to take the value zero. Hence, if we restrict attention to regular curves all these pathologies can be avoided. The definition of a 'Curve' ensures that we can always find a regular parametrization of each component. We use the word "closed" to describe components of a Curve that are like a distorted circle and so not surprisingly they have periodic parametrizations. Any component that is not like a distorted circle is like a distorted copy of the real line and so not surprisingly has a one-to-one parametrization. The circle and a branch of the hyperbola, described on page 21, are typical of closed and not closed Curves.

(ii) *Page 15: line 9 of the section* Here the Euclidean coordinate functions are real-valued functions on some interval I. When we described a vector field in terms of its Euclidean coordinate functions they were real-valued functions on E^3.

Supplementary Comments

(i) *Page 16: Example 4.2(3)*

$$\alpha(t) = (2\cos^2 t, \sin 2t, 2\sin t) = (2\cos^2 t, 2\sin t\cos t, 2\sin t).$$

Since

$$(2\cos^2 t)^2 + (2\sin t\ \cos t)^2 + (2\sin t)^2$$
$$= 4\cos^2 t(\cos^2 t + \sin^2 t) + 4\sin^2 t = 4(\cos^2 t + \sin^2 t) = 4$$

it follows that $\alpha(t)$ belongs to S.

Also

$\alpha(t) - (1,0,0) = (2\cos^2 t - 1, \sin 2t, 2\sin t) = (\cos 2t, \sin 2t, 2\sin t)$ and since $\cos^2 2t + \sin^2 2t = 1$ it follows that $\alpha(t)$ belongs to C.

As t tends to 0 the point $\alpha(t)$ tends to (2,0,0), and as t tends to $\pi/2$ the point $\alpha(t)$ tends to (0,0,2), and hence α follows the route sliced from C by the sphere S.

(ii) *Page 18: line 6* Since $\alpha_i(t) = p_i + q_i$ $(i = 1, 2, 3)$ it follows that $d\alpha_i/dt(t) = q_i$ $(i = 1, 2, 3)$ and hence $\alpha'(t) = (q_1, q_2, q_3)_{\alpha(t)}$.

(iii) *Page 19: Lemma 4.5* By the definition of the composite of two functions,

$$(\alpha(h))(s) = \alpha(h(s)) \text{ and } (\alpha_i(h))(s) = \alpha_i(h(s)).$$

Hence, since $\beta(s) = \alpha(h(s)) = (\alpha(h))(s)$,

$$\beta'(s) = (\alpha(h))'(s).$$

Since $\alpha(h(s)) = (\alpha_1(h(s)), \alpha_2(h(s)), \alpha_3(h(s)))$,

$$(\alpha(h))(s) = ((\alpha_1(h))(s), (\alpha_2(h))(s), (\alpha_3(h))(s))$$

and by Definition 4.3

$$(\alpha(h))'(s) = ((\alpha_1(h))'(s), (\alpha_2(h))'(s), (\alpha_3(h))'(s)).$$

The result now follows as on page 19.

Summary

Notation

$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$	Page 15, line 7
$\alpha = (\alpha_1, \alpha_2, \alpha_3)$	Page 15, line 10
α	Page 15, Definition 4.1
$\alpha'(t)$	Page 17, Definition 4.3
C: f = a	Page 20, line -5.

Definitions

(i)	Euclidean coordinate functions of a curve $\alpha = (\alpha_1, \alpha_2, \alpha_3)$	Page 15, line 9
(ii)	Curve α	Page 15, Definition 4.1
(iii)	Velocity vector of a curve $\alpha'(t)$	Page 17, Definition 4.3
(iv)	Reparametrization of a curve	Page 18, Definition 4.4
(v)	Periodic curve	Page 20, line 13
(vi)	Period of a curve	Page 20, line 15
(vii)	Regular curve	Page 20, line -10
(viii)	'Curves' on the plane defined implicitly $C\colon f = a$	Page 20, line -5
(ix)	Closed curve on the plane	Page 21, line 5 and Text: page 22

Examples

(i)	Straight line. $\alpha(t) = p + tq$	Page 15, Example 4.2.(1)
(ii)	Helix. $\alpha(t) = (a\cos t, a\sin t, bt)$	Page 15, Example 4.2.(1)

Results

(i)	If β is the reparametrization of α by h, then $\beta'(s) = \left(\frac{dh}{ds}\right)(s).\alpha'(h(s)).$	Page 19, Lemma 4.5
(ii)	If α is a curve in E^3 and f is a differentiable function on E^3, then $\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t).$	Page 19, Lemma 4.6

Techniques

(i)	Determination of a parametric representation of a straight line.	Page 15, Example 4.2.(1)
(ii)	Determination of the velocity vectors of a curve.	Page 17, Definition 4.3
(iii)	Determination of the effect of reparametrization on velocity vectors.	Page 19, Lemma 4.5
(iv)	Calculation of the rate of change of a function along a curve by means of the directional derivative with respect to the velocity vector.	Page 19, Lemma 4.6
(v)	Parametrization of implicitly defined 'Curves' in E^2.	Page 21.

Exercises

Technique (i)

1. Page 21, Exercise 5.

Technique (ii)

2. Page 21, Exercise 1.

Technique (iii)

3. Page 21, Exercise 3. Evaluate $\alpha'(\pi/4)$, $\beta'(1/\sqrt{2})$ and $(dh/ds)(1/\sqrt{2})$. Verify Lemma I.4.5 in this case.

   (Note: $\sin^{-1}$ was called arc sin in M231 and $(\sin^{-1})'(s) = 1/(1-s^2)^{\frac{1}{2}}$.)

Technique (iv)

4. Page 21, Exercise 7.

Technique (v)

5. Page 22, Exercise 10.

Solutions

1. Page 21, Exercise 5.

Suppose the line is given by $\alpha(t) = p + tq$. If it passes through the points p_1 and p_2 we can assume $\alpha(0) = p_1$ and $\alpha(1) = p_2$. Then $p = p_1$ and $q = p_2 - p_1$ and hence $\alpha(t) = p_1 + t(p_2 - p_1)$. If $p_1 = (1,-3,-1)$ and $p_2 = (6,2,1)$ the curve is

$$\begin{aligned}\alpha(t) &= (1,-3,-1) + t((6,2,1) - (1,-3,-1))\\ &= (1,-3,-1) + t(5,5,2)\\ &= (1+5t, -3+5t, -1+2t).\end{aligned}$$

Similarly the line which passes through $(-1,1,0)$ and $(-5,-1,-1)$ is given by

$$\beta(s) = (-1-4s, 1-2s, -s).$$

If the lines meet, there are real numbers t and s such that

$$\alpha(t) = \beta(s).$$

i.e.
$$\begin{aligned}1+5t &= -1-4s\\ -3+5t &= 1-2s\\ -1+2t &= -s.\end{aligned}$$

The first two equations can be solved uniquely giving $t = 2$, $s = -3$, which is also a solution of the third equation, so the lines meet at $\alpha(2) = \beta(-3) = (11,7,3)$.

2. Page 21, Exercise 1.

$\alpha(t) = (2\cos^2 t,\ \sin 2t,\ 2\sin t)$.

Hence by Definition I.4.3,

$$\alpha'(t) = \left(\frac{d}{dt}(2\cos^2 t),\ \frac{d}{dt}(\sin 2t),\ \frac{d}{dt}(2\sin t)\right)_{\alpha(t)}$$

$$= (-4\cos t \sin t,\ 2\cos 2t,\ 2\cos t)_{(2\cos^2 t,\ \sin 2t,\ 2\sin t)}$$

and

$$\alpha'\left(\frac{\pi}{4}\right) = (-2, 0, \sqrt{2})_{(1, 1, \sqrt{2})}.$$

3. Page 21, Exercise 3.

By Definition I.4.4 the reparametrized curve is given by

$$\beta(s) = \alpha(h(s)) = \alpha(\sin^{-1} s).$$

Now $\alpha(t) = (2\cos^2 t,\ \sin 2t,\ 2\sin t)$ and to make progress we express $\alpha(t)$ as a function of sin t.

We find that $\alpha(t) = (2(1 - \sin^2 t),\ 2\sin t\,(1 - \sin^2 t)^{\frac{1}{2}}, 2\sin t)$, using the identities $\cos^2 t + \sin^2 t = 1$ and $\sin 2t = 2\sin t \cos t$.

Hence $\beta(s) = \alpha(\sin^{-1}(s))$

$$= (2(1 - s^2),\ 2s(1 - s^2)^{\frac{1}{2}},\ 2s).$$

The function $\sin^{-1}$ has domain $[-1, 1]$ and so is well defined for $0 < s < 1$ and takes values in the interval $0 < s < \pi/2$.

Now

$$\alpha'(t) = (-4\cos t \sin t,\ 2\cos 2t,\ 2\cos t)$$

and

$$\alpha'\left(\frac{\pi}{4}\right) = \left(-4.\frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}},\ 0,\ 2.\frac{1}{\sqrt{2}}\right) = (-2, 0, \sqrt{2}).$$

Also

$$\beta'(s) = \left(-4s,\ 2(1 - s^2)^{\frac{1}{2}} - \frac{2s^2}{(1 - s^2)^{\frac{1}{2}}}, 2\right)$$

and

$$\beta'\left(\frac{1}{\sqrt{2}}\right) = (-2\sqrt{2}, 0, 2).$$

Finally

$$\frac{dh}{ds}(s) = \frac{1}{(1 - s^2)^{\frac{1}{2}}} \quad \text{and} \quad \frac{dh}{ds}\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}.$$

Hence

$$\beta'\left(\frac{1}{\sqrt{2}}\right) = \frac{dh}{ds}\left(\frac{1}{\sqrt{2}}\right).\alpha'\left(\frac{\pi}{4}\right) \text{ and, since } h\left(\frac{1}{\sqrt{2}}\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4},$$

this agrees with Lemma I.4.5.

4. Page 21, Exercise 7.

Let $\alpha(t) = (t, 1 + t^2, t)$,

$\beta(t) = (\sin t,\ \cos t,\ t)$,

$\gamma(t) = (\sinh t,\ \cosh t,\ t)$.

Then $\alpha'(0) = \beta'(0) = \gamma'(0) = (1, 0, 1)_{(0, 1, 0)}$. The curves have the same initial velocity v_p.

By Theorem I.4.6

$$v_p[f] = \frac{d}{dt}(f(\alpha))\Big|_{t=0} = \frac{d}{dt}(f(\beta))\Big|_{t=0} = \frac{d}{dt}(f(\gamma))\Big|_{t=0}$$

and since $f = x^2 - y^2 + z^2$ we have

$$\frac{d}{dt}(f(\alpha))\Big|_{t=0} = \frac{d}{dt}(t^2 - (1 + t^2)^2 + t^2)\Big|_{t=0} = \frac{d}{dt}(-1 - t^4)\Big|_{t=0} = 0,$$

$$\frac{d}{dt}(f(\beta))\Big|_{t=0} = \frac{d}{dt}(\sin^2 t - \cos^2 t + t^2)\Big|_{t=0} = 0,$$

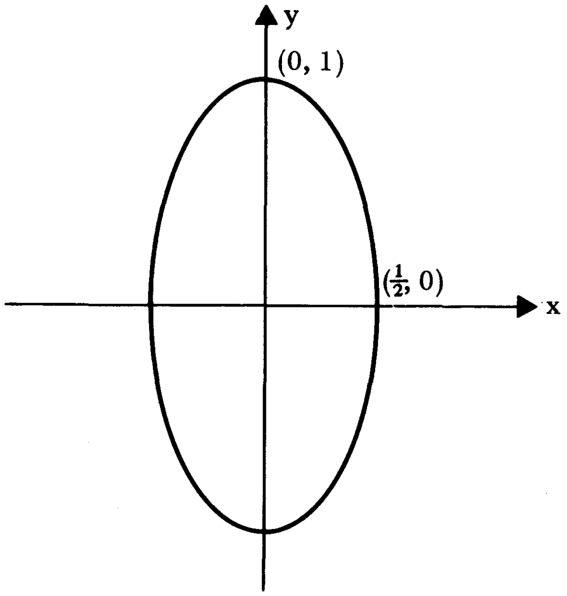
$$\frac{d}{dt}(f(\gamma))\Big|_{t=0} = \frac{d}{dt}(\sinh^2 t - \cosh^2 t + t^2)\Big|_{t=0} = 0.$$

This is the answer we expect since by Lemma I.3.2

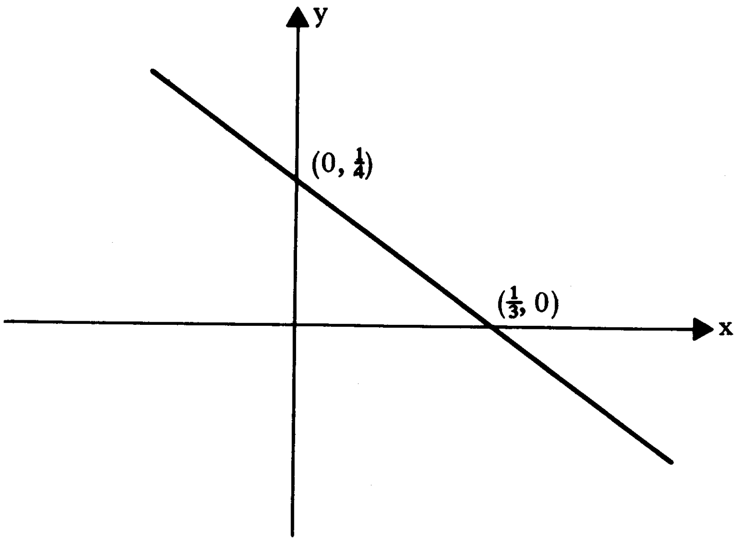
$$\begin{aligned}(1, 0, 1)_{(0, 1, 0)}[f] &= \left(1.\frac{\partial f}{\partial x} + 0.\frac{\partial f}{\partial y} + 1.\frac{\partial f}{\partial z}\right)((0, 1, 0)) \\ &= 2x((0, 1, 0)) = 0.\end{aligned}$$

5. Page 22, Exercise 10.

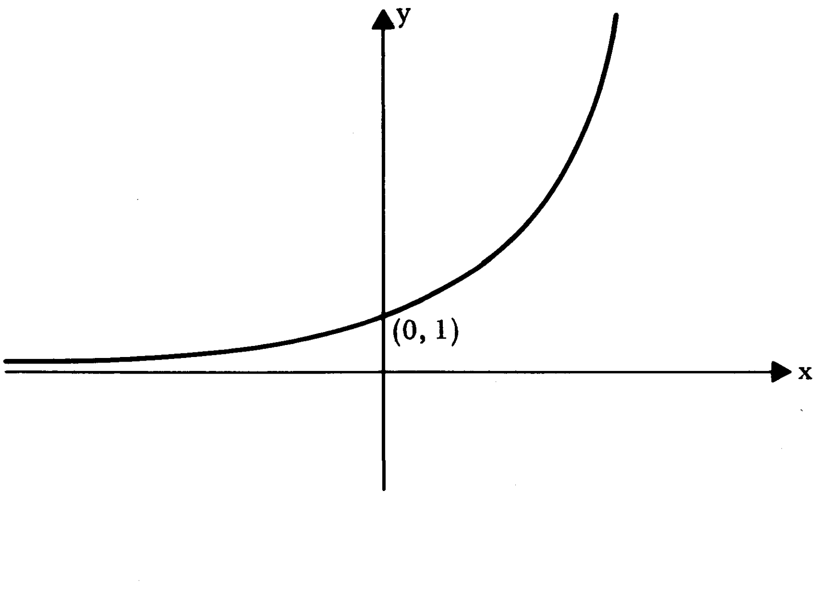
(a) $\alpha : t \longmapsto (\frac{1}{2}\cos t, \sin t)$



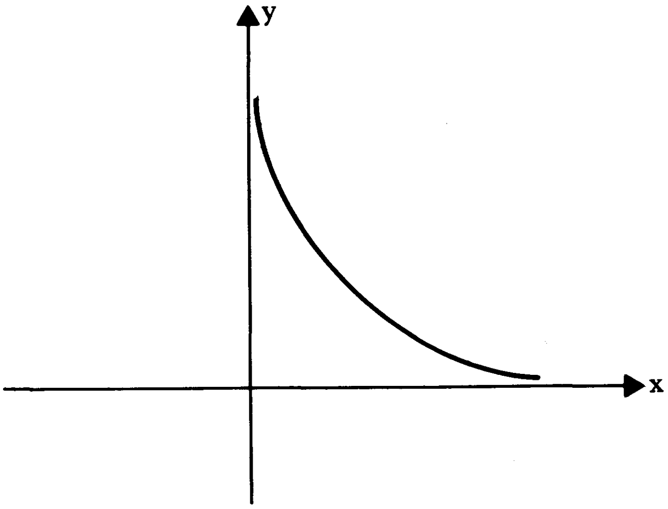
(b) $\alpha : t \longmapsto \left(t, \dfrac{1-3t}{4}\right)$



(c) $\alpha : t \longmapsto (t, e^t)$



(d) $\alpha : t \longmapsto (t, (1 - t^{\frac{2}{3}})^{\frac{3}{2}})$
$0 < t < 1$



I.5 1 -FORMS

Introduction

This section follows on from Section I.3, Directional Derivatives.

In this section we introduce 1-forms. A 1-form is an extension of the idea of linear functional. A linear functional is a linear transformation between a vector space V and the underlying vector space of reals R. Now we have a collection of vector spaces, the tangent spaces $T_p(E^3)$, and we require our 1-forms to map any tangent vector to a real number in such a way that the restriction to any one tangent space is a linear functional.

READ: Section I. 5 (pages 22-25).

Comments

(i) *Page 23: Definition 5.2* If V is a vector field, then the definition of the differential implies that $df(V) = V[f]$. This arises from the pointwise definition in the following way. For any point p in E^3

$$(df(V))(p) = df(V(p)) = V(p)[f] = (V[f])(p)$$

and so $df(V) = V[f]$.

(ii) *Page 23: Example 5.3.(1)* The symbol δ_{ij} is a very useful tool and identities such as

$$\sum_j v_j\delta_{ij} = v_1\delta_{i1} + v_2\delta_{i2} + v_3\delta_{i3} = v_i \qquad (i = 1, 2, 3)$$

occur frequently. To check this identity we look at an example, say i = 2. Then

$$\sum_j v_j\delta_{2j} = v_1\delta_{21} + v_2\delta_{22} + v_3\delta_{23}$$

$$= v_1.0 + v_2.1 + v_3.0 = v_2.$$

(iii) *Page 23: line -1* We use the same symbol for a real number and for the constant function taking that value. Hence we can write $dx_i(U_j) = \delta_{ij}$. We obtain this result from first principles as follows

$$dx_i(U_j) = U_j[x_i] = \frac{\partial x_i}{\partial x_j} = \delta_{ij}.$$

This is the basic result to remember. It can be used to obtain the results of both Example (1) and (2) as follows:

(1) $$dx_i(v_p) = dx_i\left(\sum_j v_j U_j(p)\right) = \sum_j v_j dx_i(U_j(p))$$

$$= \sum_j v_j\delta_{ij} = v_i.$$

(2) $$\psi(v_p) = \left(\sum_i f_i dx_i\right)\left(\sum_j v_j U_j(p)\right)$$

$$= \sum_{i,j} f_i(p) v_j dx_i(U_j(p)) = \sum_{i,j} f_i(p) v_j \delta_{ij}$$

$$= \sum_i f_i(p) v_i.$$

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(iv) *Page 25: Lemma 5.7* This follows from the rule for the partial differentiation of a composite given in the Additional Comments to Section I.1.

Supplementary Comments

(i) *Page 23: lines 5 to 10* Two functions are equal if they take the same value at each point p of E^3.

Now

$$\begin{aligned}
(\phi(fV+gW))(p) &= \phi((fV+gW)(p)), &&\text{by the definition of the evaluation of a 1-form on a vector field,}\\
&= \phi(f(p)V(p)+g(p)W(p)), &&\text{by the pointwise definition of operations on vector fields}\\
&= f(p)\phi(V(p))+g(p)\phi(W(p)), &&\text{by the definition of a 1-form}\\
&= f(p)(\phi(V))(p)+g(p)(\phi(W))(p)\\
&= (f\phi(V)+g\phi(W))(p).
\end{aligned}$$

Hence

$$\phi(fV+gW) = f\,\phi(V) + g\,\phi(W).$$

Similarly, since

$$\begin{aligned}
((f\phi+g\psi)(V))(p) &= (f\phi+g\psi)(V(p))\\
&= f(p)\phi(V(p))+g(p)\psi(V(p))\\
&= f(p)(\phi(V))(p)+g(p)(\psi(V))(p)\\
&= (f\phi(V)+g\psi(V))(p),
\end{aligned}$$

it follows that $(f\phi+g\psi)(V) = f\phi(V)+g\psi(V)$.

(ii) *Page 23: Definition 5.2* The function df is a 1-form since

$$\begin{aligned}
df(av_p+bw_p) &= (av_p+bw_p)[f]\\
&= av_p[f]+bw_p[f], &&\text{by Theorem I.3.3.(1),}\\
&= a\,df(v_p)+b\,df(w_p).
\end{aligned}$$

(iii) *Page 24: the proof of Lemma 5.4* Since the natural frame field forms a basis, when restricted to each vector space, any 1-form that is zero on all three of these vector fields must be the zero 1-form. For j = 1, 2, 3,

$$\begin{aligned}
\left(\phi - \sum_i \phi(U_i)dx_i\right)(U_j) &= \phi(U_j) - \sum_i \phi(U_i)dx_i(U_j)\\
&= \phi(U_j) - \sum_i \phi(U_i)\delta_{ij} = \phi(U_j) - \phi(U_j) = 0.
\end{aligned}$$

Hence $\phi - \sum_i \phi(U_i)dx_i = 0$ and $\phi = \sum_i \phi(U_i)dx_i$.

(iv) *Page 24: the proof of Corollary 5.5* This can be proved directly using Lemma I.5.4.

$$\begin{aligned} df &= \sum_i df(U_i)dx_i, && \text{by Lemma I.5.4,} \\ &= \sum_i U_i[f]\, dx_i, && \text{by Definition I.5.2,} \\ &= \sum_i \frac{\partial f}{\partial x_i} dx_i. \end{aligned}$$

(v) *Page 25: the worked example* How do we obtain the first term $2xy\ dx$? Now $f = x^2y + \ldots$ and hence $df = d(x^2y) + \ldots = d(x^2)y + x^2dy + \ldots$, by Lemma I.5.6. To evaluate $d(x^2)$ rigorously we need to introduce the squaring function $S : t \longmapsto t^2$ for which $S' : t \longmapsto 2t$.

$$\begin{aligned} \text{Then } d(x^2) = d(S(x)) &= S'(x)dx, && \text{by Lemma I.5.7,} \\ &= 2x\, dx. \end{aligned}$$

Hence $df = 2xy\ dx + x^2dy + \ldots$

Finally $v_p[f] = df(v_p)$ is evaluated using the result of Example 5.3.(2).

Summary

Notation

ϕ	Page 22, Definition 5.1
$\phi(v_p)$	Page 22, Definition 5.1
ϕ_p	Page 22, line -11
$\phi + \psi$	Page 22, line -7
$f\phi$	Page 22, line -4
$\phi(V)$	Page 22, line -2
df	Page 23, Definition 5.2
$df(v_p)$	Page 23, Definition 5.2
$df(V)$	Text: page 30
dx_i	Page 23, Example 5.3.(1)
δ_{ij}	Page 23, line -12

Definitions

(i)	1-form ϕ	Page 22, Definition 5.2
(ii)	Addition of 1-forms	Page 22, line -7
(iii)	Multiplication of a 1-form by a real-valued function	Page 22, line -4
(iv)	The effect of a 1-form on a vector field	Page 22, line -2
(v)	The differential of a real valued function df	Page 22, Definition 5.2
(vi)	Kronecker delta δ_{ij}	Page 23, line -12

Results

(i)	$\phi(V)$ is linear in both the 1-forms ϕ and vector fields V.	Text: page 31
(ii)	$dx_i(v_p) = v_i$.	Page 23, Example 5.3.(1)
(iii)	If $\phi = \Sigma f_i dx_i$, then $\phi(v_p) = \Sigma f_i(p)v_i$.	Page 23, Example 5.3.(2)
(iv)	$dx_i(U_j) = \delta_{ij}$.	Page 23, line -1
(v)	$\phi = \Sigma\phi(U_i)dx_i$.	Page 24, Lemma 5.4
(vi)	$df = \Sigma \frac{\partial f}{\partial x_i} dx_i$.	Page 24, Corollary 5.5
(vii)	$d(f + g) = df + dg$.	Page 24, line -9
(viii)	$d(fg) = gdf + fdg$.	Page 24, Lemma 5.6
(ix)	If $f : E^3 \longrightarrow R$ and $h : R \longrightarrow R$ then $d(h(f)) = h'(f)df$.	Page 25, Lemma 5.7

Techniques

(i)	Recognition of a 1-form, by comparison with the formula $\phi(av_p + bw_p) = a\phi(v_p) + b\phi(w_p)$.	Page 22, Definition 5.1
(ii)	Expressing a 1-form in standard form using the formula $\phi = \Sigma\phi(U_i)dx_i$.	Page 24, Lemma 5.4
(iii)	Evaluation of the effect of a 1-form on a tangent vector using the formula $\phi(v_p) = \sum_i f_i(p)v_i$ where $\phi = \sum_i f_i dx_i$.	Page 23, Example 5.3.(2)
(iv)	Evaluation of the effect of a 1-form on a vector field using the identity $dx_i(U_j) = \delta_{ij}$ and linearity.	Text: page 30
(v)	Calculation of differentials using the 'chain rule' $df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3$.	Page 24, Corollary 5.5
(vi)	Calculation of differentials using the linear, Leibnizian and composite properties $d(f + g) = df + dg$,	Page 24, line -9
	$d(fg) = g\,df + f\,dg$,	Page 24, Lemma 5.6
	$d(h(f)) = h'(f)df$.	Page 25, Lemma 5.7
(vii)	Evaluation of directional derivatives using differentials $v_p[f] = df(v_p)$.	Page 23, Definition 5.2

Exercises

Techniques (i) and (ii)

1. Page 26, Exercise 7.

Technique (iii)

2. Page 25, Exercise 1(c).

Technique (iv)

3. Page 25, Exercise 3. Evaluate ϕ on W only.

Technique (v)

4. Page 26, Exercise 5(a).

Technique (vi)

5. Page 26, Exercise 5(b). $\left(\text{Note: } (\tan^{-1})'(t) = \dfrac{1}{1+t^2}.\right)$

Technique (vii)

6. Page 26, Exercise 6(b).

Solutions

1. Page 26, Exercise 7.

To be a 1-form, ϕ needs to be linear on each tangent space $T_p(E^3)$. That is, the mapping

$(v_1, v_2, v_3) \longmapsto \phi((v_1, v_2, v_3)_p)$ must be linear for each p in E^3.

(a) The mapping is

$$(v_1, v_2, v_3) \longmapsto v_1 - v_3,$$

which is a linear mapping from E^3 to R independent of whichever point p we choose. Hence this does give us a 1-form.

Now $\phi(U_1(p)) = \phi((1, 0, 0)_p) = 1$

$\phi(U_2(p)) = \phi((0, 1, 0)p) = 0$

$\phi(U_3(p)) = \phi((0, 0, 1)_p) = -1$

and hence, since $\phi = \sum_i \phi(U_i)dx_i$,

$$\phi = 1.dx_1 - 1.dx_3 = dx_1 - dx_3.$$

(b) This is not linear, since, for a fixed point p with $p_1 \neq p_3$ and non-zero vector v, the definition gives

$$\phi(2v_p) = \phi(v_p) = p_1 - p_3 \neq 0$$

while we would expect $\phi(2v_p) = 2\phi(v_p)$ if ϕ were linear.

(c) For fixed $p = (p_1, p_2, p_3)$ the mapping is

$$(v_1, v_2, v_3) \longmapsto p_3v_1 + p_1v_2,$$

which is linear and hence ϕ is a 1-form.

Now $\phi(U_1(p)) = \phi((1, 0, 0)_p) = p_3 = x_3(p)$ and hence $\phi(U_1) = x_3$;

$\phi(U_2(p)) = \phi((0, 1, 0)_p) = p_1 = x_1(p)$ and hence $\phi(U_2) = x_1$;

$\phi(U_3(p)) = \phi((0, 0, 1)_p) = 0$

and hence $\phi = \sum_i \phi(U_i)dx_i = x_3dx_1 + x_1dx_2$.

(d) For fixed p the mapping $v_p \longmapsto v_p[x^2+y^2]$ is linear, by Theorem I.3.3.(1), and since $v_p[x^2+y^2]$ is a real number ϕ_p is a linear functional and ϕ is a 1-form. Now by the definition of a differential

$$v_p[x^2+y^2] = d(x^2+y^2)(v_p).$$

Hence, since $\phi : v_p \longmapsto v_p[x^2+y^2] = d(x^2+y^2)(v_p)$, it follows that $\phi = d(x^2+y^2) = 2x\,dx + 2y\,dy$.

(e) This mapping is linear for each fixed p and so does define a 1-form. Obviously it is the zero 1-form, for which again we use the greatly overworked symbol 0.

(f) This fails to five us a 1-form for much the same reason as for part (b).

2. Page 25, Exercise 1(c).

If $\psi = \sum_i f_i dx_i$ then $\psi(v_p) = \Sigma f_i(p)v_i$.

Here $\psi = (z^2 - 1)dx - dy + x^2dz$ and hence

$$\psi(v_p) = (p_3^2 - 1)v_1 - v_2 + p_1^2 v_3.$$

If $v_p = (1, 2, 3)_{(0, -2, 1)}$ then $\psi(v_p) = -2$.

3. Page 25, Exercise 3.

$$\begin{aligned}\phi(W) &= (x^2dx - y^2dz)((xy + yz)U_1 - yzU_2 - xyU_3)\\ &= x^2(xy + yz)dx(U_1) - x^2yz\,dx(U_2) - x^3y\,dx\,(U_3)\\ &\quad -y^2(xy + yz)dz(U_1) + y^3z\,dz(U_2) + xy^3\,dz(U_3), \qquad \text{by linearity,}\\ &= x^2(xy + yz) + xy^3 = x^3y + x^2yz + xy^3\\ &= xy(x^2 + xz + y^2),\end{aligned}$$

since $dx(U_1) = dz(U_3) = 1$ and all the other similar terms are zero.

4. Page 26, Exercise 5(a).

By Corollary I.5.5, $df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy + \dfrac{\partial f}{\partial z}dz$.

Here $f = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ and hence $\dfrac{\partial f}{\partial x} = x(x^2 + y^2 + z^2)^{-\frac{1}{2}}$.

Using similar results for $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$ we obtain

$$df = (x^2 + y^2 + z^2)^{-\frac{1}{2}}(x\,dx + y\,dy + z\,dz).$$

5. Page 26, Exercise 5(b).

When dealing with the function $\tan^{-1}(y/x)$ we must restrict ourselves to a region of E^3 on which the function x is never zero. The largest such domain is $\{p \in E^3 : p_1 \neq 0\}$.

Now $d(\tan^{-1}(y/x)) = (\tan^{-1})'(y/x)d(y/x)$, by Lemma I.5.7,

$$= \frac{1}{1+(y/x)^2}d(y/x),$$

$$= \frac{x^2}{x^2+y^2}\left(\frac{1}{x}dy + y d\left(\frac{1}{x}\right)\right),$$ by Lemma I.5.6,

$$= \frac{x^2}{x^2+y^2}\left(\frac{dy}{x} - \frac{y\,dx}{x^2}\right),$$ by Lemma I.5.7,

$$= \frac{x\,dy - y\,dx}{x^2+y^2}.$$

6. Page 26, Exercise 6(b).
By Corollary I.5.5

$$d(xe^{yz}) = \frac{\partial(xe^{yz})}{\partial x}dx + \frac{\partial(xe^{yz})}{\partial y}dy + \frac{\partial(xe^{yz})}{\partial z}dz$$

$$= e^{yz}(dx + xz\,dy + xy\,dz).$$

Hence $df[v_p] = e^{p_2 p_3}(v_1 + p_1 p_3 v_2 + p_1 p_2 v_3)$ and if $v_p = (1, 2, 3)\ (0, -2, 1)$ it follows that $df[v_p] = e^{-2}$.

I.6 DIFFERENTIAL FORMS

Introduction

This section follows on from Section I.5, where we gave an abstract definition of a 1-form and then proved that any 1-form is a combination of the differentials of the natural coordinate functions. In this section we introduce 0-, 1-, 2- and 3-forms as formal expressions in terms of these differentials and define the wedge product and exterior derivative of such forms. In Chapter IV and Chapter VI we shall see that these higher dimensional forms can be defined in terms of multilinear functionals operating on tangent vectors.

READ: Section I.6 (pages 26-31).

Comments

(i) *Page 27: lines 15 to 19* Take these formal expressions as the definitions of 0-, 1-, 2-, 3-forms and treat the previous paragraphs as motivation only.

In certain cases it is possible to simplify the formal expression for a p-form without introducing any ambiguity.

(a) If some, but not all, of the coefficient functions are the zero functions we can omit the corresponding part of the linear combination. For example

$$e^x dx + xyz\,dy + 0dz \equiv e^x dx + xyz\,dy$$

and

$$z^2 dx\,dy + 0\,dx\,dz + 0\,dy\,dz \equiv z^2 dx\,dy.$$

(b) If all the coefficient functions are zero we have the zero p-form and when no confusion can arise we just write this as 0.

That is,

(i) the constant function with value 0 is the zero 0-form.

(ii) $0\,dx + 0\,dy + 0\,dz \equiv 0$ is the zero 1-form.

(iii) $0\,dx\,dy + 0\,dx\,dz + 0\,dy\,dz \equiv 0$ is the zero 2-form.

(iv) $0\,dx\,dy\,dz \equiv 0$ is the zero 3-form.

It is very unlikely that this abuse of the symbol 0 will ever lead to an ambiguity.

(c) If some of the coefficient functions are the constant function with value 1 we omit them. For example,

$$e^y dx + 1\,dy + 1\,dz = e^y dx + dy + dz$$

and $$e^y dx\,dy + 0\,dx\,dy + 1\,dy\,dz = e^y dx\,dy + dy\,dz.$$

(d) If some of the coefficient functions are the constant function with value -1 we omit the symbol "1". For example,

$$1\,dx\,dy + (-1)\,dx\,dz + 0dy\,dz \equiv dx\,dy - dx\,dz$$

and $$(-1)\,dx\,dy\,dz \equiv -dx\,dy\,dz.$$

More generally, if some coefficient function is $-f$, for some more commonly used function f, we write $\ldots -f \ldots$ instead of $\ldots + (-f) \ldots$. For example,

$$x\,dx + (-y)\,dy + 0\,dz \equiv x\,dx - y\,dy.$$

(ii) *Page 27: line 23* Using the interpretation of 1-forms given in the last section we can prove that to add 1-forms or multiply them by some real-valued function all we have to do is to add the coefficient functions or multiply them by the given function. That is,

$$\Sigma f_i dx_i + \Sigma g_i dx_i = \Sigma(f_i + g_i)dx_i,$$

$$f(\Sigma f_i dx_i) = \Sigma(ff_i)dx_i.$$

Since we have no way of interpreting 2- and 3-forms yet the best we can do is to define addition and scalar multiplication in a similar way. That is

$$(f_1\,dx\,dy + g_1\,dx\,dz + h_1\,dy\,dz) + (f_2\,dx\,dy + g_2\,dx\,dz + h_2\,dy\,dz)$$

$$= (f_1 + f_2)dx\,dy + (g_1 + g_2)dx\,dz + (h_1 + h_2)dy\,dz;$$

$$k(f\,dx\,dy + g\,dx\,dy + h\,dy\,dz) = kf\,dx\,dy + kg\,dx\,dz + kh\,dy\,dz;$$

$$f_1\,dx\,dy\,dz + f_2\,dx\,dy\,dz = (f_1 + f_2)dx\,dy\,dz;$$

$$g(f\,dx\,dy\,dz) = gf\,dx\,dy\,dz.$$

(iii) *Page 27: Example 6.1* This example serves as a definition of the wedge product.

The product is given by the following procedure.

(a) Expand using the distributive law, bringing the coefficient functions to the front, dropping the symbol $\wedge$ but maintaining the order of the differentials.

(b) Drop terms with a repeated differential.

(c) Reorder the strings of differentials using the alternation rule.

(iv) *Page 28: Definition 6.3* The wedge products $df_i \wedge dx_i$ are expanded using Corollary I.5.5.

That is,

$$df_i \wedge dx_i = \left(\frac{\partial f_i}{\partial x_1} dx_1 + \frac{\partial f_i}{\partial x_2} dx_2 + \frac{\partial f_i}{\partial x_3} dx_3\right) \wedge dx_i$$

$$= \frac{\partial f_i}{\partial x_1} dx_1 dx_i + \frac{\partial f_i}{\partial x_2} dx_2 dx_i + \frac{\partial f_i}{\partial x_3} dx_3 dx_i.$$

The definition of the exterior derivative can be extended to cover 2-forms. If

$$\eta = f\,dx\,dy + g\,dx\,dz + h\,dy\,dz$$

then

$$d\eta = df \wedge dx\,dy + dg \wedge dx\,dz + dh \wedge dy\,dz$$

and this expression is again expanded using Corollary I.5.5.

(v) *Page 29: line 13* Though we can multiply 1-forms by any real-valued functions, in this formula the coefficients a and b are only real numbers or equivalently constant functions.

(vi) *Page 30: lines 11 and 12* Since dx dy dz is obtained from dz dx dy by two interchanges the expression is multiplied by $(-1)^2 = 1$.

Summary

Notation

$f\,dx + g\,dy + h\,dz$	Page 27, line 16
$f\,dx\,dy + g\,dx\,dz + h\,dy\,dz$	Page 27, line 18
$f\,dx\,dy\,dz$	Page 27, line 19
$\phi\wedge\psi$	Page 27, Example 6.1
$d\phi$	Page 28, Definition 6.3

Definitions

(i)	Differential form 0-form 1-form 2-form 3-form	Page 27, lines 15-19
(ii)	Addition and multiplication of forms	Page 27, lines 20-23
(iii)	Wedge product $\phi\wedge\psi$	Page 27, Example 6.1
(iv)	Exterior derivative $d\phi$	Page 28, Definition 6.3 and Text: page 38

Results

(i)	The wedge product of 1-forms satisfies the alternation rule $\phi\wedge\psi = \psi\wedge\phi$	Page 28, Lemma 6.2
(ii)	The exterior derivative satisfies linear and Leibnizian properties $d(a\phi + b\psi) = a\,d\phi + b\,d\psi$ $d(fg) = df\,g + f\,dg$ $d(f\phi) = df\wedge\phi + f\,d\phi$ $d(\phi\wedge\psi) = d\phi\wedge\psi - \phi\wedge d\psi.$	Page 29, line 13 and Theorem 6.4.

Techniques

(i)	Addition and scalar multiplication of differential forms.	Page 27, lines 20-23
(ii)	Expansion of the wedge product of differential forms.	Page 27, Example 6.1
(iii)	Expansion of the exterior derivative.	Page 28, Definition 6.3
(iv)	Expansion of the exterior derivatives using the linear and Leibnizian properties.	Page 29, Theorem 6.4

Exercises

Technique (i)

1. (a) Add the 2-forms

$$x\,dx\,dy + y\,dx\,dz \text{ and } -x\,dx\,dy + e^{xy}dx\,dz + 2dy\,dz.$$

(b) Multiply the second of the above 2-forms by the function e^{-xy}.

Technique (ii)

2. Form the wedge product of

(a) $yz\,dx + dz$ and $dy + z\,dz$,

(b) $yz\,dx + dz$ and $-x\,dx\,dy + e^{xy}dx\,dz + 2dy\,dz$.

Technique (iii)

3. Find the exterior derivative of

(a) $xyz\,dx + dz$,

(b) $xyz\,dx\,dy$.

Technique (iv)

4. Page 31, Exercise 3.

Theory Exercise

5. Page 31, Exercise 7.

Solutions

1. (a)

$$\begin{aligned}&(x\,dx\,dy + y\,dx\,dz) + (-x\,dx\,dy + e^{xy}dx\,dz + 2dy\,dz)\\&= (x + (-x))dx\,dy + (y + e^{xy})dx\,dz + 2dy\,dz\\&= 0\,dx\,dy + (y + e^{xy})dx\,dz + 2dy\,dz\\&= (y + e^{xy})dx\,dz + 2dy\,dz.\end{aligned}$$

(b) $e^{-xy}(-x\,dx\,dy + e^{xy}dx\,dz + 2dy\,dz)$

$= -xe^{-xy}dx\,dy + e^{xy}e^{-xy}dx\,dz + 2e^{-xy}dy\,dz$

$= -xe^{-xy}dx\,dy + 1\,dx\,dz + 2e^{-xy}dy\,dz$

$= -xe^{-xy}dx\,dy + dx\,dz + 2e^{-xy}dy\,dz.$

2. (a) $(yz\,dx + dz) \wedge (dy + z\,dz)$

$= yz\,dx\,dy + yz^2\,dx\,dz + dz\,dy + z\,dz\,dz,$ by the distributive rule,

$= yz\,dx\,dy + yz^2\,dx\,dz + dz\,dy,$ since $dzdz = 0$,

$= yz\,dx\,dy + yz^2\,dx\,dz - dy\,dz,$ by the alternation rule.

(b) $(yz\,dx + dz) \wedge (-x\,dx\,dy + e^{xy}dx\,dz + 2dy\,dz)$

$= -xyz\,dx\,dx\,dy + yze^{xy}dx\,dx\,dz + 2yz\,dx\,dy\,dz$

$-x\,dz\,dx\,dy + e^{xy}dz\,dx\,dz + 2dz\,dy\,dz,$ by the distributive law,

$= 2yz\,dx\,dy\,dz + x\,dz\,dx\,dy,$ since $dx\,dx\,dy = dx\,dx\,dz$

$= dz\,dx\,dz = dz\,dy\,dz = 0,$

$= (2yz - x)dx\,dy\,dz,$ since $dz\,dx\,dy = dx\,dy\,dz.$

3. (a) $d(xyz\,dx + dz) = d(xyz\,dx + 1\,dz)$

$= d(xyz) \wedge dx + d(1) \wedge dz$

$= d(xyz) \wedge dx,$

since the differential $d(1)$, of the constant function with value 1, is the zero 1-form and since $0 \wedge dz$ is the zero 2-form,

$= (yz\,dx + xz\,dy + xy\,dz) \wedge dx,$

by Corollary I.5.5,

$= yz\,dx\,dx + xz\,dy\,dx + xy\,dz\,dx,$

by the distributive rule,

$= xz\,dy\,dx + xy\,dz\,dx,$

since $dx\,dx = 0$,

$= -xz\,dx\,dy - xy\,dx\,dz,$

by the alternation rule.

(b) $d(xyz\,dx\,dy) = d(xyz) \wedge dx\,dy$

$= (yz\,dx + xz\,dy + xy\,dz) \wedge dx\,dy$

$= yz\,dx\,dx\,dy + xz\,dy\,dx\,dy + xy\,dx\,dx\,dy$

$= xy\,dz\,dx\,dy$

$= xy\,dx\,dy\,dz.$

4. Page 31, Exercise 3.

By Corollary I.5.5, $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$.

By Definition I.6.3, $d(df) = \sum_i d\left(\frac{\partial f}{\partial x_i}\right) \wedge dx_i$.

Again using Corollary I.5.5 we have

$$d(df) = \sum_i \left(\sum_j \frac{\partial^2 f}{\partial x_i\,\partial x_j}\; dx_j\right) \wedge dx_i = \sum_i \sum_j \frac{\partial^2 f}{\partial x_i\,\partial x_j}\; dx_j dx_i.$$

There are nine terms in this sum, which we must show is zero. We know that the terms involving $dx_1 dx_1$, $dx_2 dx_2$ and $dx_3 dx_3$ are already zero. The remaining six terms occur in pairs such as

$$\frac{\partial^2 f}{\partial x_1\,\partial x_2}\; dx_2 dx_1 \;+\; \frac{\partial^2 f}{\partial x_2\,\partial x_1}\; dx_1 dx_2.$$

By the alternation rule this is

$$\left(-\frac{\partial^2 f}{\partial x_2\,\partial x_1} + \frac{\partial^2 f}{\partial x_1\,\partial x_2}\right) dx_1 dx_2.$$

But we know that $\partial^2 f/\partial x_1 \partial x_2 = \partial^2 f/\partial x_2 \partial x_1$ for suitably differentiable functions, and so the remaining terms in the sum for d(df) cancel in pairs.

Hence $d(df) = 0$.

By Theorem I.6.4,

$$\begin{aligned} d(f\,dg) &= df \wedge dg + f\,d(dg), \\ &= df \wedge dg, \qquad \text{by the above result.} \end{aligned}$$

5. Page 31, Exercise 7.

Any 1-form ϕ is of the form

$$\phi = \sum_i f_i dx_i,$$

where f_i and x_i are differentiable functions.

By Definition I.6.3,

$$d\varphi = \sum_i df_i \wedge dx_i$$

and by Theorem I.6.4 (3),

$$d(d\phi) = \sum_i (d(df_i) \wedge dx_i - df_i \wedge d(dx_i)).$$

By Exercise 3 we know that both $d(df_i)$ and $d(dx_i)$ are zero and so

$$d(d\phi) = 0.$$

I.7 MAPPINGS

Introduction

This section follows on from Sections I.3 and I.4.

In Section I.3 we introduced directional derivatives in order to differentiate functions from E^3 to R, and in Section I.4 we introduced the componentwise differentiation of functions from R to E^3. In this section we combine these ideas in order to differentiate functions from E^n to E^m, for any positive integers n and m. The derivative mappings of these functions turn out to be linear transformations between corresponding tangent spaces and we shall describe the matrices representing them.

READ: Section I.7 (pages 32-39).

Comments

(i) *Section I.7* In this section there are several straightforward generalizations of definitions that have previously been applied to E^3 only. Those that are not specifically mentioned are:

(a) Natural coordinate functions $x_1, \dots, x_n$, which generalize those of Definition I.1.2. If p is a point in E^n then $x_i(p) = p_i$ $(i = 1, 2, \dots, n)$.

(b) Tangent vectors v_p in E^n, which generalize those of Definition I.2.1. The tangent vector v_p represents the "arrow" from p to p + v in E^n.

(c) Curves $\alpha(t) = (\alpha_1(t), \dots, \alpha_n(t))$ in E^n, which generalize those of Definition I.4.1.

(d) Straight lines $t \longmapsto p + tv$ in E^n.

(e) The velocity vector $\alpha'(t) = ((d\alpha_1/dt)(t), \dots, (d\alpha_n/dt)(t))_{\alpha(t)}$ for a curve α in E^n, which generalizes Definition I.4.3.

(f) The directional derivative $v_p[f]$ of a function f on E^n with respect to a tangent vector v_p, which generalizes that of Definition I.3.1. That is, $v_p[f] = (d/dt)\,(f(p + tv))\big|_{t=0}$.

(ii) *Page 36: line 3*

$$\begin{aligned}
F_*(v) &= F(p + tv)'(0) \\
&= ((p_1 + tv_1)^2 - (p_2 + tv_2)^2, 2(p_1 + tv_1)(p_2 + tv_2))'(0) \\
&= (2(p_1 + tv_1)v_1 - 2(p_1 + tv_2)v_2, 2(p_1 + tv_1)v_2 + 2v_1(p_2 + tv_2))(0) \\
&= 2(p_1v_1 - p_2v_2, p_1v_2 + p_2v_1).
\end{aligned}$$

(iii) *Page 37: following Corollary 7.7* To obtain the j-th column of a matrix representing a linear transformation with respect to given bases, we find the coordinates for the image of the j-th basis vector in the domain, in terms of the basis for the codomain.

In Example 7.3(2) we take as basis vectors for the domain $U_1(p)$ and $U_2(p)$ and for the codomain $U_1(F(p))$ and $U_2(F(p))$. Hence, by Corollary 7.7, the linear transformation is represented by

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial u}(p) & \dfrac{\partial f_1}{\partial v}(p) \\ \\ \dfrac{\partial f_2}{\partial u}(p) & \dfrac{\partial f_2}{\partial v}(p) \end{pmatrix}$$

where $f_1 = u^2 - v^2$ and $f_2 = 2uv$.

All such matrices are called *Jacobian matrices.*

Hence at $p = (p_1,p_2)$ the Jacobian matrix is

$$\begin{pmatrix} 2u(p) & -2v(p) \\ 2v(p) & 2u(p) \end{pmatrix} = \begin{pmatrix} 2p_1 & -2p_2 \\ 2p_2 & 2p_1 \end{pmatrix}$$

The image of a typical vector $v = (v_1,v_2)$ under this transformation is

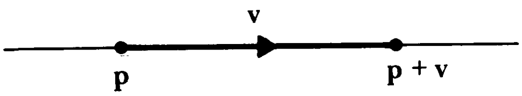
$$\begin{pmatrix} 2p_1 & -2p_2 \\ 2p_2 & 2p_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2 \begin{pmatrix} p_1v_1 - p_2v_2 \\ p_2v_1 - p_1v_2 \end{pmatrix}$$

which is the answer obtained on page 36, line 3. Note that while dealing with matrices we represent tangent vectors by column vectors.

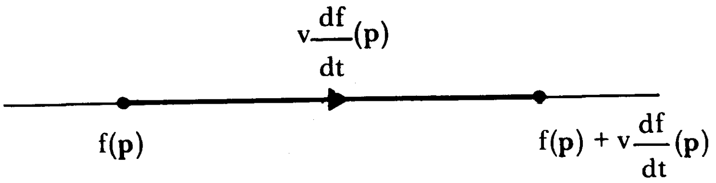
Additional Text

(a) Let f be a mapping from R to R. Then the Jacobian matrix is the 1×1 matrix (df/dt), where here we do not need partial derivatives since f is a function of t only.

Now a tangent vector in R, v_p is an arrow from p to p + v,



and this is mapped by f_{*p} to the tangent vector $v(df/dt)(p)$ at $f(p)$. That is it is multiplied in length by $(df/dt)(p)$.



(b) Suppose we have a curve $\alpha : R \longrightarrow R^3$, where α has coordinate functions $(\alpha_1, \alpha_2, \alpha_3)$, i.e.

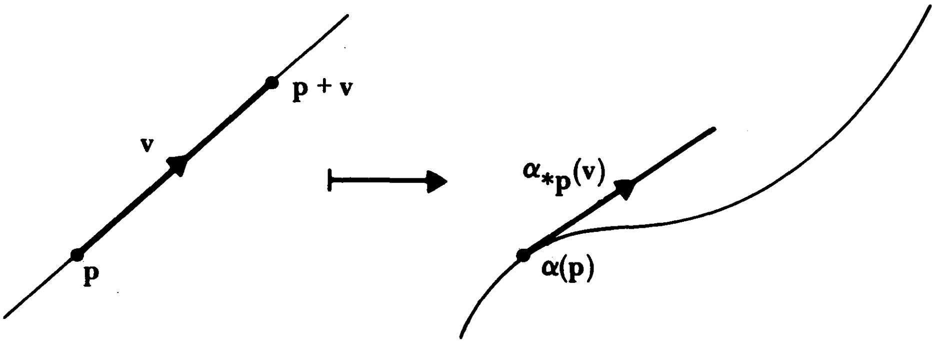
$$\alpha : t \longmapsto \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)).$$

Then we obtain the 3 X 1 Jacobian matrix

$$\begin{pmatrix} \dfrac{d\alpha_1}{dt} \\ \dfrac{d\alpha_2}{dt} \\ \dfrac{d\alpha_3}{dt} \end{pmatrix}$$

A tangent vector v, at p, is mapped to the tangent vector $\alpha_{*p}(v)$, at $\alpha(p)$. Reverting to row vectors, this is the tangent vector

$$\left(v\frac{d\alpha_1}{dt}(p),\quad v\frac{d\alpha_2}{dt}(p),\quad v\frac{d\alpha_3}{dt}(p)\right)_{\alpha(p)}$$



(c) Suppose we have a real-valued function $F : E^3 \longrightarrow R$, then the Jacobian matrix is the 1 X 3 matrix

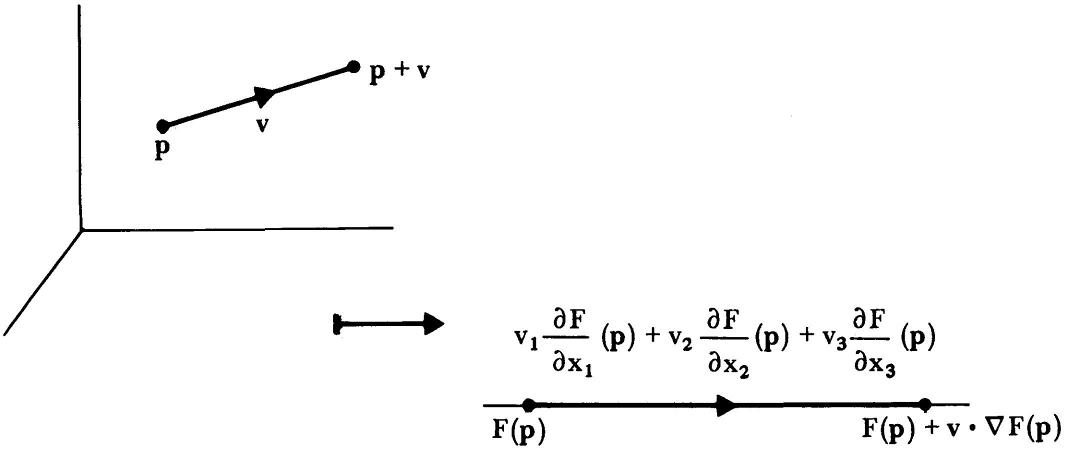
$$\left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \frac{\partial F}{\partial x_3}\right).$$

If $v_p = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_p$ is a typical tangent vector, it is mapped by F_{*p}

to the tangent vector

$$\left(\frac{\partial F}{\partial x_1}(p), \frac{\partial F}{\partial x_2}(p), \frac{\partial F}{\partial x_3}(p)\right)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ at } F(p) \text{ in } R.$$

This is $v_1 \ (\partial F/\partial x_1)(p) + v_2 \ (\partial F/\partial x_2)(p) + v_3 \ (\partial F/\partial x_3)(p)$, which we have already met as the directional derivative $v_p[F]$.


p + v
v
p
$v_1 \frac{\partial F}{\partial x_1}(p) + v_2 \frac{\partial F}{\partial x_2}(p) + v_3 \frac{\partial F}{\partial x_3}(p)$
F(p)
F(p) + v • ∇F(p)

(d) *The Composite Rule* Suppose we have two mappings $F : E^n \longrightarrow E^m$ and $G : E^m \longrightarrow E^r$, then we can form the composite mapping $G \circ F : E^n \longrightarrow E^r$. The obvious question is whether we can calculate the derivative map $(G \circ F)_*$ in terms of the derivatives G_* and F_*. The answer is yes. We state the result, which should be intuitively obvious, and investigate some of its consequences.

Theorem

For mappings $F : E^n \longrightarrow E^m$ and $G : E^m \longrightarrow E^r$

$$(G \circ F)_* = G_* \circ F_*.$$

To be more precise we need to take into consideration the point at which each of the derivatives is evaluated.

Now

$$F_{*p} : T_p(E^n) \longrightarrow T_{F(p)}(E^m)$$

$$G_{*F(p)} : T_{F(p)}(E^m) \longrightarrow T_{G \circ F(p)}(E^r)$$

$$(G \circ F)_{*p} : T_p(E^n) \longrightarrow T_{G \circ F(p)}(E^r)$$

and hence $(G \circ F)_{*p} = G_{*F(p)} \circ F_{*p}$. (This may remind you of the composite rule for ordinary differentiation – $D(g \circ f) = (Dg \circ f).Df$.)

Considering all points p of the domain at the same time we can write the composite rule as

$$(G \circ F)_* = G_{*F} \circ F_*.$$

This form of the composite rule accounts for all of the composite rules that we have encountered. When we expand it in terms of Jacobian matrices the composition on the right hand side then stands for the usual matrix multiplication.

(i) Given maps $f : R \longrightarrow R$ and $g : R \longrightarrow R$, the composite rule gives the form of the chain rule encountered in M100.

(ii) Given a map $h : R \longrightarrow R$ followed by $\alpha : R \longrightarrow E^3$, as in Lemma I.4.5, the composite rule gives

$$(\alpha \circ h)_{*s} = \alpha_{*h(s)} \circ h_{*s}.$$

Now $h_* = \dfrac{dh}{ds}$ and α_* is represented by $\begin{pmatrix} \dfrac{d\alpha_1}{dt} \\ \dfrac{d\alpha_2}{dt} \\ \dfrac{d\alpha_3}{dt} \end{pmatrix}$

Hence if we write $\beta = \alpha \circ h$ the composite rule gives

$$\begin{pmatrix} \dfrac{d\beta_1}{dt}(s) \\ \dfrac{d\beta_2}{dt}(s) \\ \dfrac{d\beta_3}{dt}(s) \end{pmatrix} = \begin{pmatrix} \dfrac{d\alpha_1}{dt}(h(s)) \\ \dfrac{d\alpha_2}{dt}(h(s)) \\ \dfrac{d\alpha_3}{dt}(h(s)) \end{pmatrix} \cdot \frac{dh}{ds}(s)$$

i.e. $\beta'(s) = \dfrac{dh}{ds}(s)\alpha'(s)$.

(iii) Given a map $f : E^3 \longrightarrow R$ followed by $h : R \longrightarrow R$, as in Lemma I.5.7, we can write the composite rule as

$$(h \circ f)_* = h_{*f} \circ f_*.$$

Now h_* is just the derivative which can be written h', and so $h_{*f} = h'(f)$. Hence in terms of Jacobian matrices

$$\left(\frac{\partial(h(f))}{\partial x_1}, \frac{\partial(h(f))}{\partial x_2}, \frac{\partial(h(f))}{\partial x_3} \right) = h'(f)\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right),$$

i.e. $$\frac{\partial(h(f))}{\partial x_i} = h'(f)\frac{\partial f}{\partial x_i} \qquad (i = 1,2,3).$$

(iv) Given a map $\alpha : R \longrightarrow E^3$ followed by a map $f : E^3 \longrightarrow R$, as in Lemma I.4.6, then the composite rule states

$$(f \circ \alpha)_{*t} = f_{*\alpha(t)} \circ \alpha_{*t}.$$

In terms of Jacobian matrices, this becomes

$$\frac{d(f(\alpha))}{dt}(t) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\alpha(t)), & \frac{\partial f}{\partial x_2}(\alpha(t)), & \frac{\partial f}{\partial x_3}(\alpha(t)) \end{pmatrix} \begin{pmatrix} \frac{d\alpha_1}{dt}(t) \\ \frac{d\alpha_2}{dt}(t) \\ \frac{d\alpha_3}{dt}(t) \end{pmatrix}$$

so

$$\frac{d(f(\alpha))}{dt}(t) = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}(\alpha(t)) \frac{d\alpha_i}{dt}(t).$$

Now read the proof of Lemma I.4.6 and Lemma I.3.2, where $\alpha : t \longmapsto p + tv$, so that $(d\alpha_i/dt)(0) = (d/dt)(p_i + tv_i)\big|_{t=0}\, v_i$.

Summary

Notation

$F = (f_1, \ldots, f_m)$	Page 33, Definition 7.1
F_*	Page 35, Definition 7.4
$\begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}$	Page 37 and Text, page 44

Definitions

(i)	Euclidean coordinate functions $(f_1, \ldots, f_m)$	Page 33, Definition 7.1
(ii)	Image of curve under a mapping $F(\alpha)$	Page 33, Definition 7.2
(iii)	The derivative map F_*	Page 35, Definition 7.4
(iv)	Jacobian matrix	Page 37 and Text, page 44

Results

(i)	The derivative map can be described in terms of the coordinate functions and directional derivatives $F_*(v) = (v[f_1], \ldots, v[f_m])$.	Page 36, Theorem 7.5
(ii)	The derivative map $F_{*p} : T_p(E^n) \longrightarrow T_{F(p)}(E^m)$ is a linear transformation.	Page 36, Corollary 7.6
(iii)	The effect of F_* on the standard basis is $F_*(U_j) = \sum_i \frac{\partial f_i}{\partial x_j} \overline{U}_i$.	Page 37, Corollary 7.7

(iv) The derivative map preserves velocities.

If $\beta = F(\alpha)$ then $\beta' = F_*(\alpha')$. Page 38, Theorem 7.8

(v) The composite rule:

If $F : E^n \longrightarrow E^m$ and $G : E^m \longrightarrow E^r$ then

$(G \circ F)_{*p} = G_{*F(p)} \circ F_{*p}$.

Techniques

(i) Description of mappings in terms of Euclidean coordinate functions. Page 33, Definition 7.1

(ii) Pictorial description of simple mappings.

(iii) Determination of derivative maps,

  (a) from first principles Page 35, Definition 7.4

  (b) from directional derivatives of coordinate functions Page 36, Theorem 7.5

  (c) from Jacobian matrix. Text, page 44

(iv) Calculation of the derivative mapping of a composite using the composite rule:

$$(G \circ F)_* = G_{*F} \circ F_*$$

where if the derivatives are represented by Jacobian matrices the composition on the right is matrix multiplication.

Exercises

Technique (i)

1. Express the following mappings from E^3 to E^3 in terms of Euclidean coordinate functions

  (a) $F : p \longmapsto -3p$,

  (b) $F : p \longmapsto (e^{p_1 p_2}, p_3 + 2p_1, p_1^2)$.

Technique (ii)

2. Page 39, Exercise 2.

Technique (iii)

3. (a) Page 39, Exercise 4.

  (b) Find $F_*(v_p)$, using directional derivatives, for $F = (x \cos y, x \sin y, z)$, $v = (2, -1, 3)$ and $p = (0, 0, 0)$.

  (c) Describe the derivative mappings for F given in Example (1), page 34.

Technique (iv)

4. Use the composite rule for Jacobian matrices to find $(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$ for the composite mapping in Exercise 4(a), page 6.

Solutions

1. (a) Since $p \longmapsto -3p$,

$$F(p) = (-3p_1, -3p_2, -3p_3)$$
$$= (-3x(p), -3y(p), -3z(p))$$

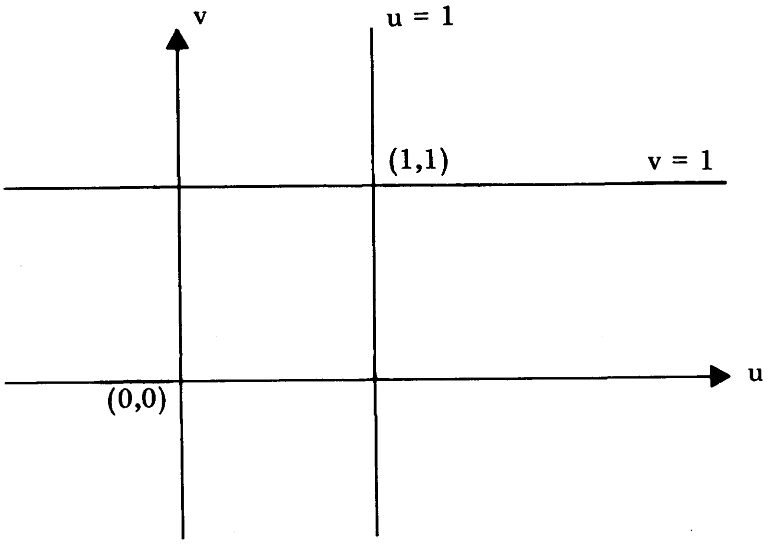
and so, in terms of Euclidean coordinate functions,

$$F = (-3x, -3y, -3z).$$

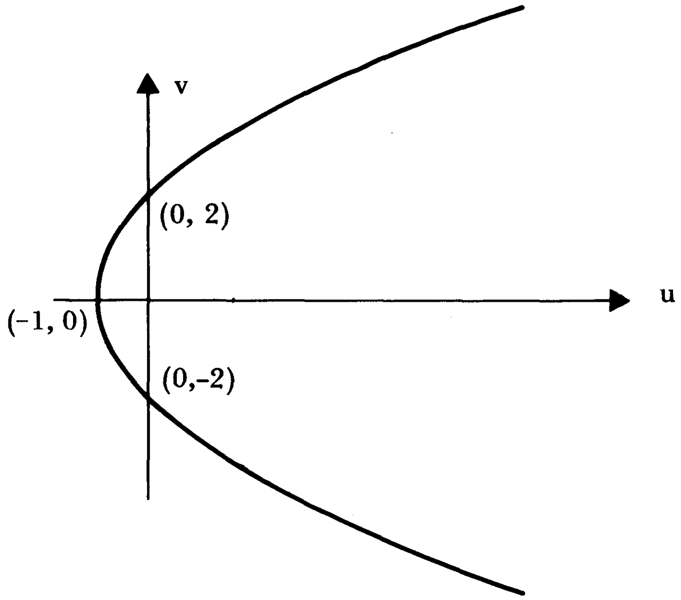
(b) $$F(p) = (e^{p_1 p_2}, p_3 + 2p_1, p_1^2)$$
$$= (e^{x(p)y(p)}, z(p) + 2x(p), (x(p))^2)$$
$$= (e^{xy}(p), (z + 2x)(p), x^2(p))$$

and so $F = (e^{xy}, z + 2x, x^2)$.

2. Page 39, Exercise 2.
The lines $u = 1$, and $v = 1$ are the sets of points p such that $u(p) = p_1 = 1$ and $v(p) = p_2 = 1$. These are shown on the following diagram.



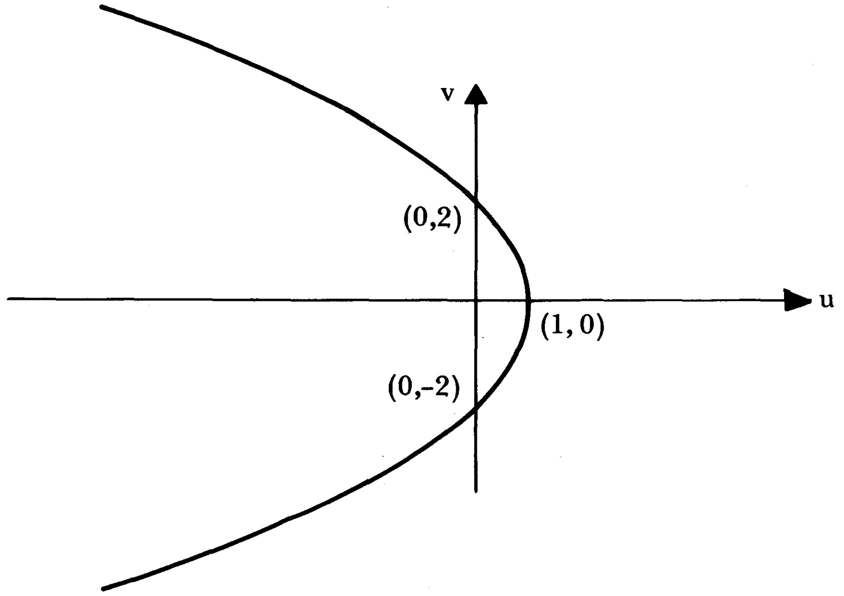
A point on the horizontal line is of the form $(t, 1)$ and is mapped by F to $(t^2 - 1, 2t)$. So, for instance, $(0, 1) \longmapsto (-1, 0)$, $(1, 1) \longmapsto (0, 2)$ and $(-1, 1) \longmapsto (0, -2)$.



How do we describe such a curve? A point $p = (p_1, p_2)$ is on the curve if $p_1 = t^2 - 1, p_2 = 2t$ for some t. From these identities we deduce that $4(p_1 + 1) = p_2^2$. Using the notation of Section I.4 we write

$$p \in M : 4(u + 1) - v^2 = 0.$$

The line $u = 1$ consists of points of the form $(1, t)$. These are mapped by F to $(1 - t^2, 2t)$. This is a typical point on the parabola $v^2 = -4(u - 1)$, which is shown on the following diagram.



3. (a) Page 39, Exercise 4(a).

$$F_*(v_p) = \frac{d}{dt}(F(p + tv))\Big|_{t=0} \text{ at } F(p)$$

$$= \frac{d}{dt}(F(p_1 + tv_1, p_2 + tv_2, p_3 + tv_3))\Big|_{t=0}$$

$$= \frac{d}{dt}(p_1 + tv_1 - p_2 - tv_2, p_1 + tv_1 + p_2 + tv_2, 2p_3 + 2tv_3)\Big|_{t=0}$$

$$= (v_1 - v_2, v_1 + v_2, 2v_3) \text{ at } F(p).$$

Page 39, Exercise 4(b).

If F is linear then $F(p + tv) = F(p) + tF(v)$.

$$\text{Hence } F_*(v_p) = \frac{d}{dt}(F(p + tv))\big|_{t=0}, \text{ at } F(p),$$

$$= \frac{d}{dt}(F(p) + tF(v))\big|_{t=0}, \text{ at } F(p),$$

$$= F(v) \text{ at } F(p),$$

so $F_*(v_p) = F(v)_{F(p)}$.

(b) If $F = (x\cos y,\ x\sin y,\ z)$, then by Theorem I.7.5

$F_*(v_p) = (v_p[x\cos y],\ v_p[x\sin y],\ v_p[z])_{F(p)}$.

Now if $v = (2,-1,3)$ and $p = (0,0,0)$, then by Lemma I.3.2

$$\begin{aligned} v_p[x\cos y] &= 2\frac{\partial}{\partial x}(x\cos y)(p) - \frac{\partial}{\partial y}(x\cos y)(p) + 3\frac{\partial}{\partial z}(x\cos y)(p) \\ &= 2\cos y(0,0,0) + x\sin y(0,0,0) + 3.0(0,0,0) \\ &= 2\cos 0 + 0\sin 0 + 3\cdot 0 \\ &= 2. \end{aligned}$$

Similarly,

$$\begin{aligned} v_p[x\sin y] &= 2\sin y(0,0,0) - x\cos y(0,0,0) + 0 \\ &= 2\sin 0 - 0\cos 0 = 0. \\ v_p[z] &= 2.0 - 1.0 + 3.1 \\ &= 3. \end{aligned}$$

So

$$F_*(v_p) = (2,0,0)_{(0\cos 0,\ 0\sin 0,\ 0)} = (2,0,3)_{(0,0,0)}.$$

(c) If $F = (x - y, x + y, z)$ the Jacobian matrix for F_* is

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This maps the vector v to

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 - v_2 \\ v_1 + v_2 \\ v_3 \end{pmatrix}$$

and so $F_*(v_p) = (v_1 - v_2, v_1 + v_2, v_3)_{F(p)}$, as expected.

4. We have f described as a composite of functions, $f = h \circ g$, where $g : p \longmapsto (g_1(p), g_2(p), g_3(p))$. That is, g_1, g_2, g_3 are the coordinate functions for g and so in this example

$$g = (x + y, y^2, x + z).$$

Since $f : E^3 \longrightarrow R$, $g : E^3 \longrightarrow E^3$ and $h : E^3 \longrightarrow R$, the corresponding Jacobian matrices are

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right),$$

$$\begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \\ \frac{\partial g_3}{\partial x} & \frac{\partial g_3}{\partial y} & \frac{\partial g_3}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2y & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

and $\left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}\right) = (2x, -z, -y)$.

Now by the composite rule $f_{*p} = h_{*g(p)} \circ g_{*p}$

so

$$\left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p)\right) = \Big(2x(g(p)), -z(g(p)), -y(g(p))\Big) \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2y(p) & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \Big(2g_1(p), -g_3(p), -g_2(p)\Big) \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2p_2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \left(2p_1 + 2p_2, -p_1 - p_3, -p_2^2\right) \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2p_2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= (2p_1 + 2p_2 - p_2^2, 2p_1 + 2p_2 - 2p_1p_2 - 2p_2p_3, -p_2^2),$$

and hence

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x + 2y - y^2, 2x + 2y - 2xy - 2yz, -y^2).$$

FURTHER EXERCISES AND SOLUTIONS

Section I.1

Technique Exercises

Pages 5-6, Exercises 1(b), 2(a), 2(d), 4(b), 4(c).

Solutions

1(b). $xy(2y + x)\sin z$

2(a). 0

2(d). $t^4(1 - t^3)$

4(b). $\dfrac{\partial f}{\partial x} = 2e^{2x}(1 - e^y)$

4(c). $\dfrac{\partial f}{\partial x} = 4x.$

Section I.2

Technique Exercise

Page 11, Exercise 4.

Other Recommended Exercise

Page 11, Exercise 5. This extends the concepts of linear independence and linear combinations to cover vector fields.

Solutions

4. $f = kx^2$, $g = -ky^2$ for any real function k on R^3,

e.g. $f = x^2$, $g = -y^2$.

5(a). $V_1(p) = (1,0,-p_1)_p$

$V_2(p) = (0,1,0)_p$

$V_3(p) = (p_1,0,1)_p$

and since $\begin{vmatrix} 1 & 0 & p_1 \\ 0 & 1 & 0 \\ p_1 & 0 & 1 \end{vmatrix} = 1 + p_1^2 \neq 0$ these vector fields are linearly independent.

5(b). $$(xU_1 + yU_2 + zU_3)(p) = (p_1,p_2,p_3)_p$$
$$= \frac{p_1(1 - p_3)}{(1 + p_1^2)} V_1(p) + p_2V_2(p) + \frac{(p_1^2 + p_3)}{(1 + p_1^2)} V_3(p)$$

and hence

$$xU_1 + yU_2 + zU_3 = \frac{x(1 - z)}{(1 + x^2)} V_1 + yV_2 + \frac{(x^2 + z)}{(1 + x^2)} V_3.$$

Section I.3

Technique Exercises

Pages 14-15, Exercises 1(a), (b), 2(a), (b), 3(a)-(d), (f).

Solutions

1(a) and 2(a). 0

1(b) and 2(b). 896

3(a). y^3

3(b). $-3xz^2$

3(c). $yz^2(y^2z - 3x^2)$

3(d). $-yz^2(y^2z - 3x^2)$

3(f). 0

Section I.4

Technique Exercise

Page 21, Exercise 8.

Other Recommended Exercises

Page 21, Exercise 4. This exercise shows that a curve is determined uniquely by its initial point and velocity vectors.

Page 21, Exercise 6. This theory exercise deals with the definition of the directional derivative.

Page 21, Exercise 9. This deals with the geometrical interpretation of tangent lines.

Solutions

8. $$\beta(s) = \left(s, \frac{1}{s}, \sqrt{2}\log s\right),$$

$$\beta'(s) = \left(1, -\frac{1}{s^2}, \frac{\sqrt{2}}{s}\right),$$

$$\alpha'(h(s)) = \left(s, -\frac{1}{s}, \sqrt{2}\right).$$

$\frac{dh}{ds}(s) = \frac{1}{s}$ and hence $\beta'(s) = \frac{dh}{ds}(s).\alpha'(h(s))$.

4. $$\alpha(t) = (1 + \frac{t^3}{3}, \frac{t^2}{2}, e^t - 6).$$

6. Let α be any curve with $\alpha'(0) = v_p$.

Then $\frac{d(f(\alpha))}{dt}(0) = \alpha'(0)[f] = v_p[f]$.

9. (a) $u \longmapsto (2, 2u, u)$.

(b) $u \longmapsto (\sqrt{2}(1 - u), \sqrt{2}(1 + u), \frac{\pi}{4} + u)$.

Section I.5

Technique Exercises

Pages 25-26, Exercises 1(a), (b), 3(a), (c), 4, 6(a), (c).

Other Recommended Exercises

Page 25, Exercise 2. This expresses the operation of a 1-form on a vector field in terms of Euclidean coordinate functions.

Page 26, Exercise 8. This deals with the Leibnizian property of d.

Page 26, Exercises 9, 10. These explore the relationship between the differential of a function and its maxima and minima.

Page 26, Exercise 11. This deals with the relationship between the differential and the difference operator.

Solutions

1(a). 4

1(b). -4

3(a). $x^3 + x^2(1 + z) - xy^2 - \frac{y^2 z}{x}$

4(a). $5f^4 df$

4(b). $\frac{df}{2\sqrt{f}}$

4(c). $\frac{2f\,df}{1 + f^2}$

6(a). $df = y^2 dx + (2xy - z^2)dy - 2yz\,dz$, -10.

6(c). $df = \cos(xy)\cos(xz)(y\,dx + x\,dy) - \sin(xy)\sin(xz)(z\,dx + x\,dz)$, -2.

2. $\phi(V) = \sum_i \sum_j f_i v_j \, dx_i(U_j) = \sum_i \sum_j f_i v_j \delta_{ij} = \sum_i f_i v_i$.

8.
$$\begin{aligned} d(fg)(v_p) &= v_p[fg] = v_p[f]g + f v_p[g] \\ &= df(v_p)\, g + f\, dg(v_p) \\ &= (g\,df + f\,dg)(v_p) \end{aligned}$$

and hence $d(fg) = g\,df + f\,dg$.

9. $$df(v_p) = \frac{\partial f}{\partial x}(p)v_1 + \frac{\partial f}{\partial y}(p)v_2 + \frac{\partial f}{\partial z}(p)v_3$$

and hence $df(v_p) = 0$ for all v if and only if

$$\frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = \frac{\partial f}{\partial z}(p) = 0.$$

Here $\frac{\partial f}{\partial x} = -2xy, \frac{\partial f}{\partial y} = 1 - x^2 - 2yz, \frac{\partial f}{\partial z} = 1 - y^2$

and the critical points are $\pm (0, 1, \tfrac{1}{2})$.

10. If p is a local maximum or minimum then $t = 0$ is a local maximum or minimum of the function $t \longmapsto f(p + tv)$.

Hence $df[v_p] = v_p[f] = \frac{d}{dt}(f(p + tv))\big|_{t=0} = 0$.

11(a). By Taylor's Theorem

$$f(p + tv) = f(p) + t\frac{d}{dt}(f(p + tv))\big|_{t=0} + \text{Remainder}$$

$$= f(p) + t\,df(v_p) + R.$$

Hence $f(p + v) - f(p) \sim df(v_p)$.

11(b). Exact: $-0{\cdot}420$

Approximate: $df(v_p) = \left(\frac{2xy}{z}dx + \frac{x^2}{z}dy - \frac{x^2y}{z^2}dz\right)(-0{\cdot}1, 0{\cdot}1, 0{\cdot}2)_{(1,\,1\cdot5,\,1)}$

$= -\frac{1}{2}$.

Section I.6

Technique Exercises

Page 31, Exercises 1, 2, 4(a), (b), (d), 6.

Other Recommended Exercises

Page 31, Exercise 5. This expresses the wedge product of three 1-forms in terms of the determinant of their Euclidean coordinates.

Page 31, Exercise 8. This exercise deals with the relationship between the exterior derivative and the vector operations div, grad and curl.

Solutions

1(a). $yz\cos z\,dx\,dy - \sin z\,dx\,dz - \cos z\,dy\,dz$,

$\sin z\,dx\,dy + z\sin z\,dx\,dz + z\cos z\,dy\,dz$,

$-yz\,dx\,dy - yz^2\,dx\,dz + dy\,dz$.

1(b). $-z\,dx\,dy - y\,dx\,dz$,

$-\cos z\,dx\,dz + \sin z\,dy\,dz$,

0.

2. $d\phi = \frac{1}{y^2}\,dx\,dy$, $d\psi = -\,dy\,dz$,

$\phi \wedge \psi = \frac{z}{y}\,dx\,dy$ and

$d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi = \frac{1}{y}\,dx\,dy\,dz$.

4(a). $d(f\,dg + g\,df) = d(d(fg)) = 0$.

4(b). $(df - dg) \wedge (df + dg) = 2\,df \wedge dg$.

4(d). $(1 - f)df \wedge dg$

6. $dx\,dy\,dz = r\,dr\,d\vartheta\,dz$

5. Expand both the determinant and the triple wedge product in terms of the Euclidean coordinate functions. Both sides give

$$(f_{11}f_{22}f_{33} - f_{11}f_{23}f_{32} + f_{12}f_{23}f_{31} - f_{12}f_{21}f_{33} + f_{13}f_{21}f_{32} - f_{13}f_{22}f_{31})dx\,dy\,dz.$$

8(a). $df = \Sigma \frac{\partial f}{\partial x_i}\,dx_i \xleftrightarrow{(1)} \Sigma \frac{\partial f}{\partial x_i}\,U_i = \text{grad } f.$

8(b). If $V = \Sigma f_i\,U_i$ then $\phi = \Sigma f_i\,dx_i$,

$$d\phi = \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 dx_2 + \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\right) dx_1 dx_3 + \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2 dx_3$$

and $d\phi \xleftrightarrow{(2)}$ curl V.

8(c). If $V = \Sigma f_i U_i$ then $\eta = f_3 dx_1 dx_2 - f_2 dx_1 dx_3 + f_1 dx_2 dx_3$.

$d\eta = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) dx_1\,dx_2\,dx_3$ and hence $d\eta \xleftrightarrow{(2)}$ (div V)$dx\,dy\,dz$.

Section I.7

Technique Exercise

Page 39, Exercise 1.

Other Recommended Exercises

Page 40, Exercises 5 and 6. These describe the derivative of a mapping in terms of the differentials of its Euclidean coordinate functions.

Page 40, Exercise 8. This deals with the definition of the derivative.

Page 40, Exercise 9. The derivative mapping preserves directional derivatives.

Page 40, Exercise 11. This deals with inverse mappings, which you can read about in Exercise 10.

Page 40, Exercises 7 and 12. These deal with the derivative of a composite mapping.

Solutions

1(a). $(0, 0)$.

1(b). $(-3,1), (3,-1)$.

1(c). $(0, 0), (1, 0)$.

5. $F_* = (\cos y\, dx - x \sin y\, dy, \sin y\, dx + x \cos y\, dy, dz)$

(a) $(2,0,3)_{(0,0,0)}$

(b) $(2,2,3)_{(0,2,\pi)}$

6. The Jacobian matrix at p is

$$\begin{pmatrix} \cos p_2 & \sin p_2 & 0 \\ -p_1 \sin p_2 & p_1 \cos p_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This has determinant p_1 and hence F is not regular at points for which $p_1 = 0$.

8. Let α be any curve such that $\alpha'(0) = v_p$, then if $\beta = F(\alpha)$ it follows that

$$\beta'(0) = F_*(\alpha'(0)) = F_*(v_p).$$

9. Let $\alpha : t \longmapsto p + tq$.

$$R \xrightarrow{\alpha} E^n \xrightarrow{F} E^m \xrightarrow{g} R$$

$$v_p[g(F)] = \alpha'(0)[g(F)] = \frac{d}{dt}(g(F(\alpha)))\big|_{t=0}.$$

$$F_*(v_p) = (F(\alpha))'(0).$$

$$F_*(v_p)[g] = (F(\alpha))'(0)[g] = \frac{d}{dt}(g(F(\alpha)))\big|_{t=0}.$$

Hence $v_p[g(F)] = F_*(v_p)[g]$.

11(a). $F^{-1} = (v, ue^{-v})$. This is a diffeomorphism.

11(b). $F^{-1} = (u^{\frac{1}{3}}, v + u^{\frac{1}{3}})$. This is not a diffeomorphism since it is not differentiable when $u = 0$.

11(c). $F^{-1} = \left(\dfrac{(9 - u - 2v)}{2}, 5 - u - v\right)$. This is a diffeomorphism.

7. $GF = (g_1(f_1, f_2), g_2(f_1, f_2))$.

12(a). $GF = (g_1(f_1, \dots, f_m), \dots, g_p(f_1, \dots, f_m))$

12(b). $(GF)_*(\alpha') = ((GF)(\alpha))'$, by Theorem I.7.8,

$=(G(F(\alpha)))'$

$=G_*((F(\alpha))')$

$=G_*F_*(\alpha')$.

12(c). Since any $v_p = \alpha'(0)$ for some α it follows that $(GF)_* = G_*F_*$.

DIFFERENTIAL GEOMETRY

I **Calculus on Euclidean Space**

II **Frame Fields**

III **Euclidean Geometry**

IV **Calculus on a Surface**

V **Shape Operators**

VI **Geometry of Surfaces in E^3**

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Elementary Differential Geometry
Prepared by the Course Team

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1.1

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Set Book

Barrett O'Neill: *Elementary Differential Geometry* (Academic Press, 1966). It is essential to have this book: the course is based on it and will not make sense without it.

Bibliography

The set books for M201, M231 and MST 282 are referred to occasionally; they are useful but not essential. They are:

D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins: *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

E.D. Nering: *Linear Algebra and Matrix Theory* (John Wiley, 1970).

M. Spivak: *Calculus,* paperback edition, (W.A. Benjamin/Addison-Wesley 1973).

R.C. Smith and P. Smith: *Mechanics,* SI edition (John Wiley, 1972).

Also mentioned in this Part is:
T. Wilmore: *An Introduction To Differential Geometry* (O.U.P., 1964).

Conventions

Before starting work on this text, please read M334 Part Zero. Consult the Errata List and the Stop Press and make any necessary alterations for this chapter in the set book.

Unreferenced pages and sections denote the set book. Otherwise

O'Neill	denotes the set book;
Text	denotes the correspondence text;
KKOP	denotes *An Introduction to Linear Analysis* by D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins;
Nering	denotes *Linear Algebra and Matrix Theory by* E.D. Nering;
Spivak	denotes *Calculus* by M. Spivak;
Smith	denotes *Mechanics* by R.C. Smith and P. Smith.

References to Open University Courses in Mathematics take the form:

Unit M100 22, Linear Algebra I

Unit MST 281 10, Taylor Approximation

Unit M201 16, Euclidean Spaces I: Inner Products

Unit M231 2, Functions and Graphs

Unit MST 282 1, Some Basic Tools.

II.1 INTRODUCTION AND DOT PRODUCT

Introduction

This whole section depends only on Sections I.1 and I.2, and may be read easily even if you are not entirely confident about the later sections of Chapter I.

It provides a revision of the basic vector concepts that you have met in MST 282 and M201. You should make sure that you are familiar with the following ideas: the *dot product, cross product, triple scalar product* and *norm* for E^3 (*Unit MST 282 1, Some Basic Tools,* Section 2); the *dot product, norm, Schwarz inequality, angle, orthogonality, orthonormal bases* and *orthonormal expansions* for general Euclidean vector spaces (*Unit M201 16, Euclidean Spaces I: Inner Products*); *orthogonal matrices* (*Unit M201 24, Orthogonal and Symmetric Transformations,* Section 1.2). You will also find it useful to revise the technique of evaluation of a 3 X 3 determinant (KKOP, page 684).

These vector ideas are revised and then applied to the tangent vector spaces $T_p(E^3)$. For example, the dot product of two tangent vectors with the same point of application is defined by

$$v_p \cdot w_p = v \cdot w.$$

We find that the usual vector space results carry over to each of the tangent spaces. Later on we shall extend these definitions to vector fields by the pointwise principle: for example, if V and W are vector fields the dot product of V and W, V·W, will be the function from E^3 to R whose value at p is V(p)·W(p).

READ: Introduction to Chapter II and Section II.1 (pages 42–48).

Comment

(i) *Page 47: the determinant and cross product* The determinant is a function from the set of all square real matrices into the real numbers. It is written by replacing the brackets of the matrix by straight lines. The rules for evaluating it on 2 X 2 and 3 X 3 matrices are as follows:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{21}\,a_{12};$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}\,a_{23} - a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23}$$

(see KKOP, page 684)

which may be written as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix}$$
$$+ a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

This last expression gives the expansion of the formal determinant for v X w as

$$U_1(p)\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} + U_2(p)\begin{vmatrix} v_3 & v_1 \\ w_3 & w_1 \end{vmatrix} + U_3(p)\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= (v_2w_3 - v_3w_2)\,U_1(p) + (v_3w_1 - v_1w_3)\,U_2(p) + (v_1w_2 - v_2w_1)\,U_3(p)$$

$$= (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)_p.$$

O'Neill assumes that you are familiar with the following properties of determinants:

(a) a determinant is linear as a function of each of its rows;

(b) interchanging two rows of a determinant multiplies its values by – 1;

(c) a determinant has value zero if and only if its rows are linearly dependent vectors.

In *Unit M 201 5, Determinants and Eigenvalues,* you learnt these properties for the *columns* of a determinant. However, in Section 1.5 of that unit you saw that the determinant of the transpose of a matrix is the same as the determinant of the original matrix, and thus all these results are true for rows as well.

By Lemma II.1.8, v X w is orthogonal to both v and w, and

$$\|v \times w\|^2 = (v\cdot v)\,(w\cdot w) - (v\cdot w)^2$$
$$= \|v\|^2\,\|w\|^2 - (v\cdot w)^2.$$

The angle ϑ between v and w is defined by

$$v\cdot w = \|v\|\|w\| \cos \vartheta$$

(see page 44), and so

$$\|v \times w\|^2 = \|v\|^2\|w\|^2 - (\|v\|\,\|w\| \cos \vartheta)^2$$
$$= \|v\|^2\|w\|^2\,(1 - \cos^2\vartheta)$$
$$= \|v\|^2\|w\|^2 \sin^2\vartheta.$$

Taking square roots,

$$\|v \times w\| = \|v\|\|w\| (\sin^2 \vartheta)^{\frac{1}{2}}.$$

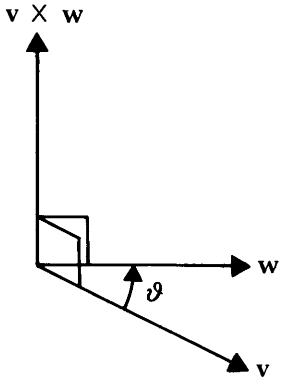
For any given value of cos ϑ (in $[-1,1]$) we can choose ϑ to lie in the range $0 \leqslant \vartheta \leqslant \pi$, that is ϑ is the smaller angle between v and w; for ϑ in this range sin ϑ is non-negative and so $(\sin^2 \vartheta)^{\frac{1}{2}} = \sin \vartheta$. Thus

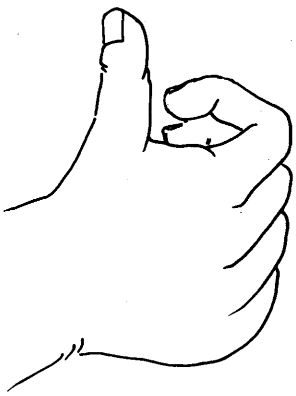
$$\|v \times w\| = \|v\|\|w\| \sin \vartheta$$

and so

$$v \times w = \|v\|\|w\| \sin \vartheta \, \mathbf{n}$$

where **n** is a unit vector orthogonal to both **v** and **w**.





In fact n is obtained from v and w by the right-hand rule (page 48, penultimate paragraph), and so O'Neill's definition of cross-product is equivalent to that given in *Unit MST 282 1, Some Basic Tools.*

One consequence of the definition of cross-product is the following.

Lemma II.1.A If e_1 and e_2 are orthogonal unit tangent vectors and $e_3 = e_1 \times e_2$ then e_3 is a unit tangent vector orthogonal to both e_1 and e_2 and

$$e_1 \times e_2 = e_3; \qquad e_2 \times e_1 = -e_3;$$

$$e_2 \times e_3 = e_1; \qquad e_3 \times e_2 = -e_1;$$

$$e_3 \times e_1 = e_2; \qquad e_1 \times e_3 = -e_2.$$

In particular, these formulas hold when $e_1 = U_1(p)$, $e_2 = U_2(p)$, in which case $e_3 = U_1(p) \times U_2(p) = U_3(p)$.

There is no reason to restrict the definition of cross product to tangent vectors. As for the dot product, we can define it for pairs of points of E^3 also. If we put $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (0, 1, 0)$, $\mathbf{u}_3 = (0, 0, 1)$ then for points $\mathbf{p}$ and $\mathbf{q}$ of E^3 we define the cross product of $\mathbf{p}$ and $\mathbf{q}$ to be

$$\mathbf{p} \times \mathbf{q} = \begin{vmatrix} u_1 & u_2 & u_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}$$

$$= (p_2q_3 - p_3q_2)u_1 + (p_3q_1 - p_1q_3)u_2 + (p_1q_2 - p_2q_1)u_3$$

$$= (p_2q_3 - p_3q_2, p_3q_1 - p_1q_3, p_1q_2 - p_2q_1).$$

Useful results about determinants and cross products are contained in the following exercises, which you should attempt now.

1. Page 49, Exercise 4.
2. Page 49, Exercise 5 (first part).
3. Page 49, Exercise 6.

Additional Text

The result of the following exercise will be needed later.

4. Page 49, Exercise 7.

Summary

Notation

$\mathbf{p}\cdot\mathbf{q}$	Page 42, Definition 1.1
$\mathbf{v_p}\cdot\mathbf{w_p}$	Page 44, Definition 1.3
$\|\mathbf{p}\|$	Page 43, line 9
$\|\mathbf{v_p}\|$	Page 44, line 12
$d(\mathbf{p}, \mathbf{q})$	Page 43, Definition 1.2
$\mathfrak{N}_\epsilon$	Page 43, line -4
${}^tA$	Page 47, line 1
$\mathbf{v_p} \times \mathbf{w_p}$	Page 47, Definition 1.7
$\mathbf{p} \times \mathbf{q}$	Text, page 8

Definitions

(i)	Dot product of points $\mathbf{p}\cdot\mathbf{q}$ of tangent vectors $\mathbf{v}\cdot\mathbf{w}$	Page 42, Definition 1.1 Page 44, Definition 1.3
(ii)	Norm of a point $\|\mathbf{p}\|$ of a tangent vector $\|\mathbf{v}\|$	Page 43, line 9 Page 44, line 12
(iii)	Euclidean distance $d(\mathbf{p}, \mathbf{q})$	Page 44, Definition 1.2

(iv)	ϵ-neighbourhood $\mathcal{N}_\epsilon$	Page 43, line -4
(v)	Open set	Page 43, line -3
(vi)	Orthogonal vectors	Page 44, line -7
(vii)	Unit vector	Page 44, line -6
(viii)	Frame, or orthonormal basis	Page 44, Definition 1.4
(ix)	Orthonormal expansion	Page 45, line -1
(x)	Attitude matrix	Page 46, Definition 1.6
(xi)	Orthogonal matrix	Page 46, line -2
(xii)	Transpose of A, ${}^tA$	Page 47, line 1
(xiii)	Cross product of points $p \times q$ of tangent vectors $v \times w$	Text, page 8 Page 47, Definition 1.7
(xiv)	Triple scalar product $u\cdot v \times w$	Page 48, line -3 of text

Results

(i)	$\|p + q\| \leqslant \|p\| + \|q\|$ (the triangle inequality) and $\|ap\| = \lvert a\rvert\, \|p\|$.	Page 43, line 11
(ii)	$\lvert v\cdot w\rvert \leqslant \|v\|\, \|w\|$ (Schwarz inequality).	Page 44, line 13
(iii)	$U_1(p), U_2(p), U_3(p)$ constitute a frame at p.	Page 45, line 6
(iv)	Orthonormal expansion. If e_1, e_2, e_3 is a frame at p and $v \in T_p(E^3)$, then $$v = (v\cdot e_1)\, e_1 + (v\cdot e_2)\, e_2 + (v\cdot e_3)\, e_3.$$	Page 45, Theorem 1.5
(v)	If e_1, e_2, e_3 is a frame, $v = \Sigma\, a_i e_i$ and $w = \Sigma\, b_i e_i$, then $$v\cdot w = a_1 b_1 + a_2 b_2 + a_3 b_3.$$	Page 46, line 7
(vi)	If A is an orthogonal matrix, ${}^tA = A^{-1}$.	Page 47, line 5
(vii)	The dot product is symmetric, bilinear, positive definite.	Page 43, lines 1-7
(viii)	The cross product is alternating, bilinear.	Page 47, line -11
(ix)	$\|v \times w\| = (v\cdot v\; w\cdot w - (v\cdot w)^2)^{\frac{1}{2}} = \|v\|\, \|w\| \sin \vartheta$.	Page 47, Lemma I.8 and Page 48, line -9
(x)	$v \times w$ is orthogonal to v and to w.	Page 47, Lemma I.8
(xi)	$v \times w = \|v\|\, \|w\| \sin \vartheta\, n$, where n is obtained by the right-hand rule.	Text, page 7
(xii)	$v \times w \neq 0$ if and only if v and w are linearly independent.	Page 49, Exercise 5
(xiii)	$u\cdot v \times w \neq 0$ if and only if u, v and w are linearly independent.	Page 49, Exercise 4(b)
(xiv)	Interchanging any two of u, v, w reverses the sign of $u\cdot v \times w$ but does not change its absolute value.	Page 49, Exercise 4(c)
(xv)	If e_1, e_2, e_3 is a frame, $$e_1 \cdot e_2 \times e_3 = \pm 1.$$	Page 49, Exercise 6

(xvi) If $\mathbf{e}_1, \mathbf{e}_2$ are orthogonal unit vectors and $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$, then $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is a frame, and

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3; \quad \mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3;$$
$$\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1; \quad \mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{e}_1;$$
$$\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2; \quad \mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2.$$

Text, page 7

(xvii) If $\mathbf{u}$ is a unit vector, any vector $\mathbf{v}$ may be expressed uniquely as

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{u})\,\mathbf{u} + \mathbf{v}_2,$$

where $\mathbf{v}_2$ is orthogonal to $\mathbf{u}$.

Page 49, Exercise 7

Techniques

(i)	Evaluation of dot and cross products and norms.	Page 42, Definition 1.1 and Page 47, Definition 1.7
(ii)	Evaluation of triple scalar products.	Page 49, Exercise 4(a)
(iii)	Expansion in terms of an orthonormal basis.	Page 45, Theorem 1.5
(iv)	Determination of the attitude matrix and use of the result ${}^tA = A^{-1}$.	Page 46, Definition 1.6

Exercises

Technique (i)

5. Page 48, Exercise 1.

Technique (ii)

6. If $\mathbf{v}$ and $\mathbf{w}$ are as in the preceding exercise and $\mathbf{u} = (-3, 1, -1)$ at the same point, compute $\mathbf{u}\cdot\mathbf{v} \times \mathbf{w}$.

Technique (iii)

7. Page 49, Exercise 3.

Technique (iv)

8. Find the attitude matrix A of the frame in the preceding exercise. What is its inverse?

Solutions

1. Page 49, Exercise 4.

(a) Suppose u, v, w are tangent vectors at the point p.

$\mathbf{u} = u_1U_1(p) + u_2U_2(p) + u_3U_3(p)$.

$\mathbf{v} \times \mathbf{w} = c_1U_1(p) + c_2U_2(p) + c_3U_3(p)$ (*)

where the c_i are found from

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} U_1(p) & U_2(p) & U_3(p) \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = c_1u_1 + c_2u_2 + c_3u_3$

which is (*) with $U_i(p)$ replaced by u_i for i = 1, 2, 3: thus replacing $U_i(p)$ by u_i in the determinant gives $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$.

(b) $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} \neq 0 \Longleftrightarrow$ the rows of the determinant are linearly independent

$\Longleftrightarrow$ u, v and w are linearly independent.

(c) Again, this follows from the properties of determinants: if any two rows in a determinant are interchanged the whole determinant changes sign.

(d) $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{u} \times \mathbf{v}$, because the dot product is symmetric,

$= \mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$, by part (c).

2. Page 49, Exercise 5.

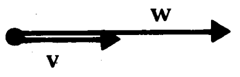
We know that $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\,\|\mathbf{w}\| \sin \vartheta$, where ϑ is the angle between v and w.

$\mathbf{v} \times \mathbf{w} = \mathbf{0} \Longleftrightarrow \|\mathbf{v} \times \mathbf{w}\| = 0$

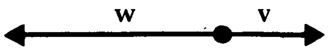
$\Longleftrightarrow \|\mathbf{v}\| = 0$ or $\|\mathbf{w}\| = 0$ or $\sin \vartheta = 0$

$\Longleftrightarrow \mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$ or ϑ is a multiple of π.

If $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$ then v and w are certainly linearly dependent. If ϑ is a multiple of π then



$\vartheta = 0$



$\vartheta = \pi$

v and w are collinear, and so linearly dependent. Conversely, if v and w are linearly dependent, either $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$ or $\sin \vartheta = 0$.

Thus $\mathbf{v} \times \mathbf{w} \neq \mathbf{0} \Longleftrightarrow$ v and w are linearly independent.

3. Page 49, Exercise 6.

$e_2 \times e_3$ is orthogonal to both e_2 and e_3. e_2 and e_3 span a plane in $T_p(E^3)$, so the tangent vectors orthogonal to both e_2 and e_3 form a line in $T_p(E^3)$. We know that e_1 is orthogonal to both e_2 and e_3 and so e_1 spans this line: thus $e_2 \times e_3$ must be a multiple of e_1.

$$\|e_2 \times e_3\| = \|e_2\| \, \|e_3\| \sin\frac{\pi}{2} = 1,$$

so

$$e_2 \times e_3 = \pm e_1.$$

Thus

$$e_1 \cdot e_2 \times e_3 = \pm e_1 \cdot e_1 = \pm 1.$$

If A is an orthogonal matrix, then it is the attitude matrix of a frame e_1, e_2, e_3, where the coordinates of e_i with respect to $U_1(p)$, $U_2(p)$, $U_3(p)$ form the ith row of A for i = 1, 2, 3. By the first part of the question, $e_1 \cdot e_2 \times e_3 = \pm 1$. However, by Exercise 4(a) on page 49,

$$e_1 \cdot e_2 \times e_3 = \begin{vmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{vmatrix} = \det A$$

so the determinant of A is ± 1.

We could have reached this last result in another way, using the facts (see *Unit M201 5, Determinants and Eigenvalues,* Section 1) that if A and B are square matrices of the same size then

$$\det({}^tA) = \det(A)$$

and

$$\det(AB) = \det(A)\det(B).$$

Now, if A is orthogonal, ${}^tAA = I$

so

$$\det({}^tAA) = \det(I) = 1$$

so

$$\det({}^tA)\det(A) = 1$$

so

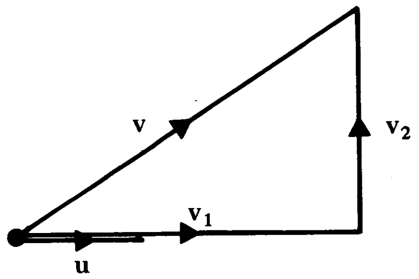
$$\det(A)\det(A) = 1$$

so

$$\det(A) = \pm 1.$$

4. Page 49, Exercise 7.

Choose $v_1 = (v \cdot u)u$, $v_2 = v - v_1$. Then certainly $v = v_1 + v_2$ and v_1 is the component of v in the u direction, by definition. Moreover, $v - v_1$ is the only possible choice for v_2 in order that $v = v_1 + v_2$, and so this expression is certainly unique.



It remains to prove just that $v_1 \cdot v_2 = 0$. Now

$$\begin{aligned} v_1 \cdot v_2 &= v_1 \cdot (v - v_1) \\ &= (v\cdot u)u \cdot (v - (v\cdot u)u) \\ &= (v\cdot u)(u\cdot v - (v\cdot u)u\cdot u) \\ &= (v\cdot u)(u\cdot v - v\cdot u), \qquad \text{because } u\cdot u = 1, \\ &= 0, \qquad \text{because } u\cdot v = v\cdot u. \end{aligned}$$

5. Page 48, Exercise 1.

(a) $v\cdot w = 1.(-1) + 2.0 + (-1).3 = -1 - 3 = -4.$

(b)

$$v \times w = \begin{vmatrix} U_1(p) & U_2(p) & U_3(p) \\ 1 & 2 & -1 \\ -1 & 0 & 3 \end{vmatrix}$$

$$= (2.3 - (-1).0, (-1).(-1) - 1.3, 1.0 - 2.(-1))$$

$$= (6, -2, 2).$$

(c) $\|v\| = (1^2 + 2^2 + (-1)^2)^{\frac{1}{2}} = \sqrt{6}$; $\dfrac{v}{\|v\|} = \dfrac{(1, 2, -1)}{\sqrt{6}}$.

$\|w\| = ((-1)^2 + 0^2 + 3^2)^{\frac{1}{2}} = \sqrt{10}$; $\dfrac{w}{\|w\|} = \dfrac{(-1, 0, 3)}{\sqrt{10}}$.

(d) $\|v \times w\| = (6^2 + (-2)^2 + 2^2) = \sqrt{44} = 2\sqrt{11}.$

Alternatively, we can use Lemma II.1.8:

$$\|v \times w\|^2 = (v\cdot v)(w\cdot w) - (v\cdot w)^2 = \|v\|^2\|w\|^2 - (v\cdot w)^2 = 6.10-(-4)^2 = 44.$$

(e) If ϑ is the angle between v and w,

$$v\cdot w = \|v\|\ \|w\| \cos\vartheta$$

so $\quad -4 = \sqrt{6}\sqrt{10}\cos\vartheta$

so $\quad \cos\vartheta = \dfrac{-4}{\sqrt{60}} = \dfrac{-4}{2\sqrt{15}} = \dfrac{-2}{\sqrt{15}}.$

6.

$$u\cdot v \times w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} -3 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & 0 & 3 \end{vmatrix}$$

$$= -3(2.3 - (-1).0) + 1((-1)^2 - 1.3) - 1(1.0 - 2.(-1))$$

$$= -3.6 + 1.(-2) - 1.2 = -22.$$

Alternatively, since $v \times w$ has already been calculated in Exercise 5, we could go straight to

$$\begin{aligned} u\cdot v \times w &= (-3, 1, -1) \cdot (6, -2, 2) \\ &= -3.6 + 1.(-2) + (-1).2 = -22. \end{aligned}$$

7. Page 49, Exercise 3.

$$e_1 \cdot e_1 = \frac{1}{6}(1^2 + 2^2 + 1^2) = 1$$

$$e_2 \cdot e_2 = \frac{1}{8}((-2)^2 + 0^2 + 2^2) = 1$$

$$e_3 \cdot e_3 = \frac{1}{3}(1^2 + (-1)^2 + 1^2) = 1$$

$$e_1 \cdot e_2 = \frac{1}{\sqrt{6}}\frac{1}{\sqrt{8}}(-2 + 0 + 2) = 0$$

$$e_2 \cdot e_3 = \frac{1}{\sqrt{8}}\frac{1}{\sqrt{3}}(-2 + 0 + 2) = 0$$

$$e_3 \cdot e_1 = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{6}}(1 - 2 + 1) = 0$$

This shows that e_1, e_2, e_3 constitutes a frame.

By Theorem II.1.5,

$$v = (v \cdot e_1)e_1 + (v \cdot e_2)e_2 + (v \cdot e_3)e_3.$$

$$v \cdot e_1 = \frac{1}{\sqrt{6}}(6 + 2 - 1) = \frac{7}{\sqrt{6}};$$

$$v \cdot e_2 = \frac{1}{\sqrt{8}}(-12 - 2) = \frac{-14}{\sqrt{8}};$$

$$v \cdot e_3 = \frac{1}{\sqrt{3}}(6 - 1 - 1) = \frac{4}{\sqrt{3}};$$

$$v = \frac{7}{\sqrt{6}}e_1 - \frac{14}{\sqrt{8}}e_2 + \frac{4}{\sqrt{3}}e_3.$$

We can check this by working out the right-hand side directly: we obtain

$$\frac{7}{\sqrt{6}}\frac{(1, 2, 1)}{\sqrt{6}} - \frac{14}{\sqrt{8}}\frac{(-2, 0, 2)}{\sqrt{8}} + \frac{4}{\sqrt{3}}\frac{(1, -1, 1)}{\sqrt{3}}$$

$$= \frac{7}{6}(1, 2, 1) - \frac{14}{8}(-2, 0, 2) + \frac{4}{3}(1, -1, 1)$$

$$= \left(\frac{7}{6}, \frac{7}{3}, \frac{7}{6}\right) + \left(\frac{7}{2}, 0, -\frac{7}{2}\right) + \left(\frac{4}{3}, -\frac{4}{3}, \frac{4}{3}\right)$$

$$= (6, 1, -1) = v.$$

8. We simply take the coordinates of e_1, e_2, e_3 and write them as the rows of A. Thus

$$A = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{8}} & 0 & \frac{2}{\sqrt{8}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Since A is an attitude matrix it is orthogonal, and so we know ${}^tA = A^{-1}$: that is, A^{-1} may be obtained from A by interchanging corresponding rows and columns.

This gives

$$A^{-1} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{8}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{8}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

II.2 CURVES

Introduction

This section depends heavily on Sections I.4 and II.1 but not at all on the later sections of Chapter I.

Here we take up the story of curves left in Section I.4, and extend the definitions and results of Section II.1 to vector fields defined on curves. We immediately introduce a way of differentiating these vector fields; in particular this gives us the *acceleration* of a curve: the velocity and acceleration turn out to be the same as those of a particle travelling along the curve, as given in *Unit MST 282 2, Kinematics,* Section I.1.

All of this is simply giving us enough technical apparatus to establish the Frenet formulas, which are a core feature of the course, in the next section.

READ: Section II.2 (pages 51–55).

Comments

(i) Pages 51–52: unit-speed reparametrization This particular type of reparametrization is very important. From now on, whenever O'Neill mentions a "unit-speed reparametrization" he means not just a reparametrization which happens to have unit speed but a reparametrization of the type given in the proof of Theorem 2.1. There is a useful symmetry in this situation. If β is a unit-speed reparametrization of α we have

$$\beta(s) = \alpha(t(s));$$

in this particular case s and t are inverse functions, and so we also have

$$\beta(s(t)) = \alpha(t):$$

that is, α is also a reparametrization of β. We shall often make use of this fact.

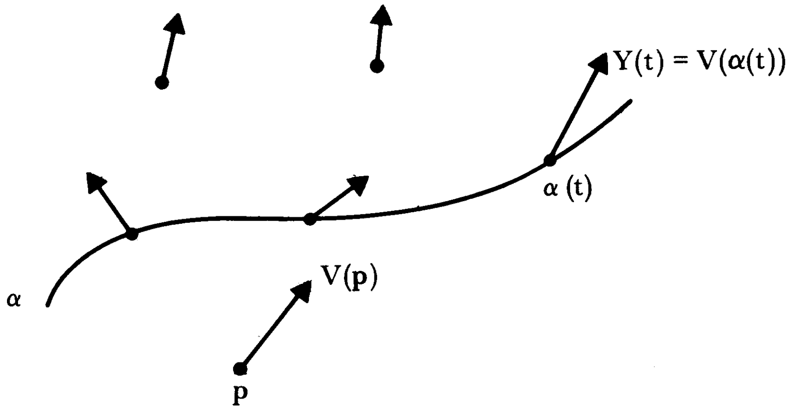
Theorem 2.1 is important because it is often sufficient to know that a unit-speed reparametrization of a curve α exists, without computing it explicitly. It is this theorem which allows many proofs to begin: "We may assume α has unit speed". This considerably simplifies many things, but you should always check that the remark is justified.

(ii) Page 52: Definition 2.2 A vector field on a curve should be distinguished from a vector field on E^3. It is true that if α is a curve and V is a vector field on E^3 then the function Y defined by

$$Y(t) = V(\alpha(t))$$

is a vector field on α (see the diagram opposite); however not every vector field on α is necessarily the restriction of a vector field on E^3. For example, it may happen that α crosses itself, that is there are different values t_1 and t_2 such that

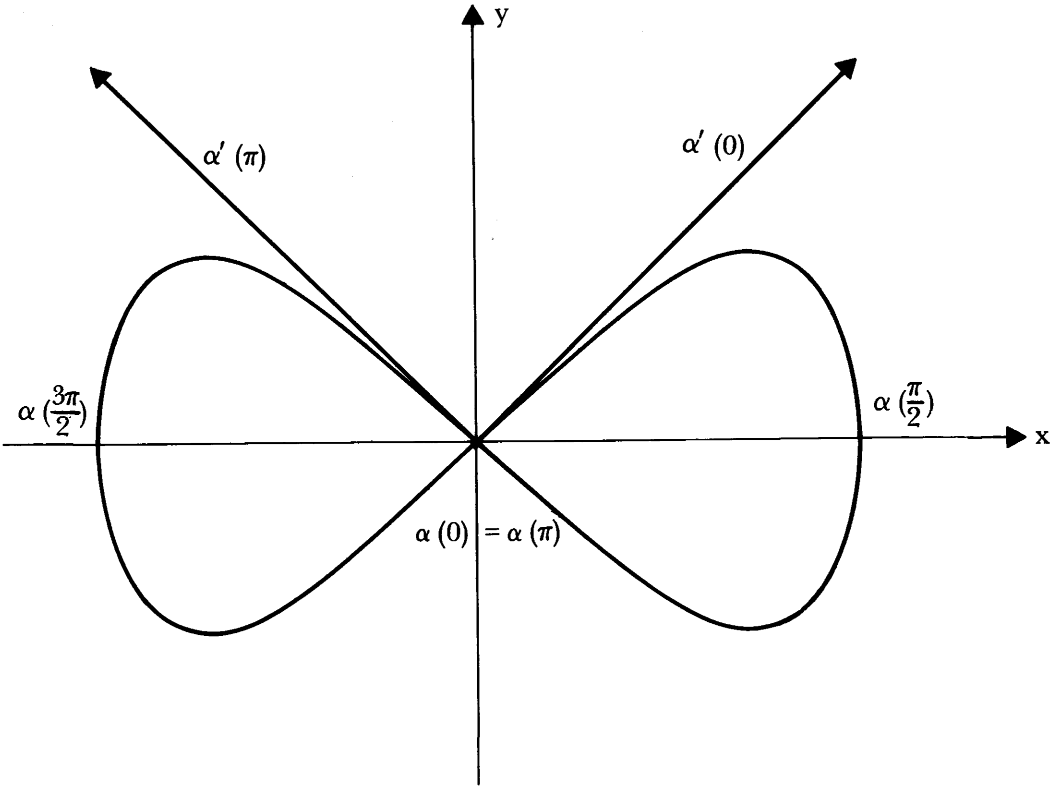
$$\alpha(t_1) = \alpha(t_2) = p;$$



if Y is a vector field on α it is not obliged to take the same values at t_1 and t_2, but if V is a vector field on E^3 then it has a unique value at p and so $V(\alpha(t_1)) = V(\alpha(t_2)) = V(p)$. A good example of this is provided by the figure-of-eight curve given by

$$\alpha(t) = (\sin t, \tfrac{1}{2}\sin 2t,\ 0) \qquad (t \in R)$$

which crosses itself at $0 = \alpha(0) = \alpha(\pi)$: at this point the vector field α' on α does take different values, for $\alpha'(0) = (1, 1, 0)$ while $\alpha'(\pi) = (-1, 1, 0)$.



(iii) Page 54: Lemma 2.3 It is important at this point to be quite clear about O'Neill's slightly unusual use of the words "constant", "straight line" and "parallel". A curve α is a *constant* if it is of the form

$$\alpha\colon t \longmapsto c$$

that is, the image of α in E^3 is a single point. A curve α is *straight line* if and only if it is of the form

$$\alpha\colon t \longmapsto p + tq$$

for some points p and q with $q \neq 0$. The image of α is certainly the line joining p and p + q. Intuitively we might think of the curves

$$\beta\colon t \longmapsto p + 2tq$$

and

$$\gamma\colon t \longmapsto p + t^3 q$$

as the same as α, because their images are also the straight line joining **p** and **p** + **q**, but it is important to realize that, under Definition I.4.1, they are different curves from α. In fact β is the reparametrization of α by the function

$$h\colon t \longmapsto 2t$$

and γ is the reparametrization of α by the function

$$k\colon t \longmapsto t^3.$$

Are β and γ straight lines? Well β is, because β can be written

$$\beta\colon t \longmapsto p + t(2q),$$

but γ cannot be written in the form

$$\gamma\colon t \longmapsto a + tb$$

for any points **a** and **b** so γ is not, technically speaking, a straight line, merely a reparametrization of one.

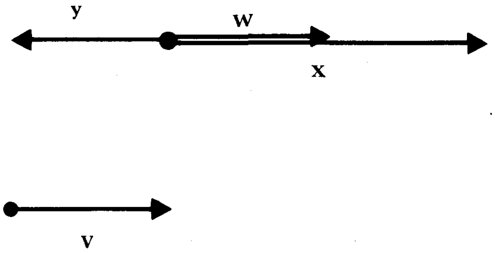
Finally, "parallel" has the meaning given on page 6, that of having the same vector parts. This is a more restrictive definition than that usually used in elementary geometry. For example, if

$$v = (1, 0, 0)_{(0, 0, 0)}$$

$$w = (1, 0, 0)_{(1, 1, 0)}$$

$$x = (2, 0, 0)_{(1, 1, 0)}$$

$$y = (-1, 0, 0)_{(1, 1, 0)}$$



then **v** is parallel to **w** but not **x** or **y**, even though they all point along the same direction. For tangent vectors at the same point we say they are *collinear* if they point along the same direction, that is if one is a scalar multiple of the other. The vectors **w**, **x** and **y** are all collinear.

(iv) Curves – a Convention From now on we adopt the convention used by Spivak that a function defined on a subset of E^n has the whole of E^n as its domain unless otherwise specified. Thus in Exercise 1 on page 55 the curve

$$\alpha(t) = (2t, t^2, \frac{t^3}{3})$$

is taken to be the curve

$$\alpha\colon t \longmapsto (2t, t^2, \frac{t^3}{3}) \qquad (t \in R).$$

Additional Text

The following exercises introduce results which will be needed later.

Attempt them now.

1. Page 56, Exercise 7.
2. Y and Z are vector fields on a curve α. Prove that $(Y \times Z)' = Y' \times Z + Y \times Z'$.

Supplementary Comment

(i) Page 51: the "standard theorem of calculus" This is a combination of several results from Spivak, which you should be able to accept.

Summary

Notation

v	Page 51, line 4
s	Page 51, line -7
Y	Page 52, Definition 2.2
α''	Page 54, line 7

Definitions

(i)	Speed v	Page 51, line 4
(ii)	Arc length	Page 51, line 11
(iii)	Arc-length function s	Page 51, line -7
(iv)	Arc-length parametrization	Page 52, line 7
(v)	Unit-speed reparametrization	Text, page 16
(vi)	Orientation-preserving reparametrization	Page 52, line -12
(vii)	Orientation-reversing reparametrization	Page 52, line -11
(viii)	Vector field (Y) on a curve	Page 52, Definition 2.2
(ix)	Euclidean coordinate functions of a vector field on a curve	Page 53, line 9
(x)	Derivative of a vector field on a curve	Page 54, line 3
(xi)	Acceleration (α'') of a curve	Page 54, line 7
(xii)	Parallel vector field	Page 54, line -10
(xiii)	Collinear tangent vectors	Text, page 18

Results

(i)	Any regular curve has a unit-speed reparametrization.	Page 51, Theorem 2.1
(ii)	If β is a unit-speed reparametrization of α, then α is a reparametrization of β.	Text, page 16
(iii)	If α has unit speed, then $s(t) = t$.	
(iv)	The linearity and Leibnizian properties of differentiation of vector fields on curves. $(aY + bZ)' = aY' + bZ'$ $(fY)' = f'Y + fY'$ $(Y \cdot Z)' = Y' \cdot Z + Y \cdot Z'$ $(Y \times Z)' = Y' \times Z + Y \times Z'$.	Page 54, lines 12-17 Text, page 19
(v)	α is constant if and only if $\alpha' = 0$.	Page 54, Lemma 2.3(1)
(vi)	α is a straight line if and only if $\alpha' \neq 0$ and $\alpha'' = 0$.	Page 55, Lemma 2.3(2)
(vii)	$\|Y\|$ is constant if and only if $Y \cdot Y' = 0$.	Page 54, lines -14 to -12
(viii)	Y is parallel if and only if $Y' = 0$.	Page 55, Lemma 2.3(3)
(ix)	$Y(h)' = h'Y'(h)$.	Page 56, Exercise 7

Techniques

(i)	Calculation of arc-length.	Page 51, line 11
(ii)	Finding an arc-length function $s = s(t)$.	Page 51, proof of Theorem 2.1
(iii)	Unit-speed reparametrization.	Page 51, Theorem 2.1
(iv)	Expression of a vector field on a curve in terms of its Euclidean coordinate functions.	Page 53, line 7
(v)	Algebra of vector fields on a curve.	Page 53, last paragraph
(vi)	Differentiation of vector fields on a curve	
	(a) from first principles	Page 54, line 3
	(b) using linearity and Leibnizian properties.	Page 54, lines 12-17 and Text, page 19
(vii)	Calculation of velocity, speed, acceleration.	Page 51, line 4 and Page 54, line 7

Exercises

Technique (i)

3. Page 55, Exercise 4. (Note that "log" means "natural logarithm".)

Techniques (ii) and (iii)

4. Page 55, Exercise 3.

Techniques (iv) and (v)

5. Page 55, Exercise 6.

Techniques (vi)(a) and (vii)

6. Page 55, Exercise 2. The required cone is the set of points in E^3 satisfying $x^2 + y^2 = z^2$.

Technique (vi)(b)

7. Let Y and Z be the vector fields on the curve $\alpha(t) = (\cos t, \sin t, t)$ given by

   $Y(t) = (\cos t, \sin t, 1)_{\alpha(t)}$,

   $Z(t) = (\sin t, t, \cos t)_{\alpha(t)}$.

   Find $Y'(t)$ and $Z'(t)$ and hence $V'(t)$ when V is each of the following:

   (a) $V(t) = Y(t) + Z(t)$
   (b) $V(t) = t^2 Y(t)$
   (c) $V = Y \cdot Z$
   (d) $V = Y \times Z$.

Theory Exercises (omit if you are short of time).

8. Y is a vector field on a curve α. Prove that if $\|Y\|$ is constant and Y' is never zero then Y'' is never orthogonal to Y. (HINT: Differentiate $Y \cdot Y$ = constant.)

9. Page 56, Exercise 9. (HINT: For the "if" part of (b), let γ be a unit-speed reparametrization of α and show that γ is a straight line by showing that γ'' is both collinear and orthogonal to γ'. By result (ii) α is a reparametrization of γ.)

Solutions

1. Page 56, Exercise 7.

   $Y(t)$ is a tangent vector at $\alpha(t)$, so $Y(h(s))$ is a tangent vector at $\alpha(h(s))$, so $Y(h)$ is a vector field on the curve $\alpha(h)$.

   We can write $Y(t) = (y_1(t), y_2(t), y_3(t))_{\alpha(t)}$

   so

   $$Y'(t) = (y'_1(t), y'_2(t), y'_3(t))_{\alpha(t)}.$$

Now

$$Y(h)(s) = (y_1(h(s)), y_2(h(s)), y_3(h(s)))_{\alpha(h(s))}$$

so

$$Y(h)'(s) = (y_1'(h(s))\, h'(s), y_2'(h(s))h'(s), y_3'(h(s))h'(s))_{\alpha(h(s))}$$

by the definition of differentiation of a vector field on a curve and by the chain rule

$$= h'(s)(y_1'(h(s)), y_2'(h(s)), y_3'(h(s)))_{\alpha(h(s))}$$

$$= h'(s)\, Y'(h(s))$$

so

$$Y(h)' = h'Y'(h).$$

Note that this result is, not surprisingly, yet another form of the chain rule!

2. If

$$Y(t) = (y_1(t), y_2(t), y_3(t))$$

and

$$Z(t) = (z_1(t), z_2(t), z_3(t))$$

then

$$(Y \times Z)\,(t) = (y_2(t)z_3(t) - y_3(t)z_2(t), y_3(t)z_1(t) - y_1(t)z_3(t), y_1(t)z_2(t) - y_2(t)z_1(t)).$$

To differentiate Y × Z we differentiate its coordinate functions. The first of these is

$$y_2z_3 - y_3z_2,$$

whose derivative is

$$y_2'z_3 + y_2z_3' - y_3'z_2 - y_3z_2',$$ by the Leibnizian rule for derivatives of functions from R to R,

$$= (y_2'z_3 - y_3'z_2) + (y_2z_3' - y_3z_2').$$

The other two coordinate functions are similar, and so we obtain

$$\begin{aligned}(Y \times Z)'(t) &= (y_2'(t)z_3(t) - y_3'(t)z_2(t), y_3'(t)z_1(t) - y_1'(t)z_3(t), \\ &\qquad y_1'(t)z_2(t) - y_2'(t)z_1(t)) \\ &\quad + (y_2(t)z_3'(t) - y_3(t)z_2'(t), y_3(t)z_1'(t) - y_1(t)z_3'(t), \\ &\qquad y_1(t)z_2'(t) - y_2(t)z_1'(t)) \\ &= Y'(t) \times Z(t) + Y(t) \times Z'(t) \\ &= (Y' \times Z + Y \times Z')(t).\end{aligned}$$

Thus

$$(Y \times Z)' = Y' \times Z + Y \times Z'.$$

3. Page 55, Exercise 4.

$\alpha(t) = (2t, t^2, \log t), t > 0.$

$\alpha(1) = (2, 1, 0) = p$

$\alpha(2) = (4, 4, \log 2) = q,$

so the curve passes through p and q.

$\alpha'(t) = (2, 2t, \frac{1}{t})_{\alpha(t)}$

$\|\alpha'(t)\| = (4 + 4t^2 + \frac{1}{t^2})^{\frac{1}{2}} = 2t + \frac{1}{t}$ since $t > 0$.

The arc-length between p and q is the arc-length from $t = 1$ to $t = 2$, which is

$$\int_1^2 \left(2t + \frac{1}{t}\right)dt = \left[t^2 + \log t\right]_1^2$$

$$= 4 + \log 2 - 1$$

$$= 3 + \log 2.$$

4. Page 55, Exercise 3.

$\alpha(t) = (\cosh t, \sinh t, t)$

$\alpha'(t) = (\sinh t, \cosh t, 1)_{\alpha(t)}$

$\|\alpha'(t)\| = (\sinh^2 t + \cosh^2 t + 1)^{\frac{1}{2}} = (2\cosh^2 t)^{\frac{1}{2}},$ because $1 + \sinh^2 t = \cosh^2 t$,

$= \sqrt{2} \cosh t.$

$$s(t) = \int_0^t \|\alpha'(u)\|\, du = \int_0^t \sqrt{2}\cosh u\, du = \sqrt{2}\sinh t$$

so

$$t(s) = \sinh^{-1}\frac{s}{\sqrt{2}}.$$

The unit-speed reparametrization of α based at $t = 0$ is

$$\beta(s) = \alpha(t(s)) = (\cosh(\sinh^{-1}\frac{s}{\sqrt{2}}), \sinh(\sinh^{-1}\frac{s}{\sqrt{2}}), \sinh^{-1}\frac{s}{\sqrt{2}})$$

$$= ((1 + \frac{s^2}{2})^{\frac{1}{2}}, \frac{s}{\sqrt{2}}, \sinh^{-1}\frac{s}{\sqrt{2}}),$$

(using the identity $\cosh^2\theta = 1 + \sinh^2\theta$ for $\theta = \sinh^{-1}\frac{s}{\sqrt{2}}$),

5. Page 55, Exercise 6.

(a) $Y(t) = 0 - \alpha(t) \quad \text{at } \alpha(t)$

$$= (-\cos t, -\sin t, -t)_{\alpha(t)}$$

$$= -\cos t\ U_1(\alpha(t)) - \sin t\ U_2(\alpha(t)) - t\ U_3(\alpha(t))$$

$$= -\cos t\ U_1 - \sin t\ U_2 - t\ U_3$$

using the convention mentioned in the middle of page 53.

(b) $\alpha'(t) = (-\sin t, \cos t, 1)_{\alpha(t)}$

$\alpha''(t) = (-\cos t, -\sin t, 0)_{\alpha(t)}$

$Y(t) = (\cos t - \sin t)U_1 + (\cos t + \sin t)U_2 + U_3.$

(c) Any tangent vector at $\alpha(t)$ orthogonal to both $\alpha'(t)$ and $\alpha''(t)$ is a scalar multiple of $\alpha'(t) \times \alpha''(t)$, if this is not zero.

$$\alpha'(t) \times \alpha''(t) = (\sin t, -\cos t, 1)_{\alpha(t)}$$

$$\|\alpha'(t) \times \alpha''(t)\|^2 = \sin^2 t + \cos^2 t + 1 = 2$$

so a unit vector in the direction of $\alpha'(t) \times \alpha''(t)$ is $\pm\ (1/\sqrt{2})\alpha'(t) \times \alpha''(t)$.

$$\text{Thus } Y(t) = \pm\left(\frac{\sin t\ U_1}{\sqrt{2}} - \frac{\cos t\ U_2}{\sqrt{2}} + \frac{U_3}{\sqrt{2}}\right).$$

(d) $Y(t) = (\alpha(t+\pi) - \alpha(t))_{\alpha(t)}$

$$= ((\cos(t+\pi), \sin(t+\pi), t+\pi) - (\cos t, \sin t, t))_{\alpha(t)}$$

$$= ((-\cos t, -\sin t, t+\pi) - (\cos t, \sin t, t))_{\alpha(t)}$$

$$= -2\cos t\ U_1 - 2\sin t\ U_2 + \pi U_3.$$

6. Page 55, Exercise 2.

Let $p = \alpha(t) = (t \cos t, t \sin t, t)$.

The usual reason for the presence of the sine and cosine functions in the Euclidean coordinate functions of a curve is to take advantage of the identity $\sin^2\theta + \cos^2\theta = 1$. Thus we see that

$$p_1^2 + p_2^2 = t^2(\cos^2 t + \sin^2 t) = t^2 = p_3^2,$$

that is, every point p on the curve satisfies

$$p_1^2 + p_2^2 = p_3^2. \qquad (*)$$

If C is the set of points satisfying (*), C intersects the plane $p_3 = r$ in the circle $p_1^2 + p_2^2 = r^2$, for each real number r. Thus C is the circular cone with axis the z-axis, vertex the origin, and half-angle $\frac{\pi}{4}$. Every point of the curve lies on C; the vertex (0, 0, 0) is $\alpha(0)$.

$$\alpha'(t) = (\cos t - t \sin t, \sin t + t \cos t, 1)_{\alpha(t)}$$

$$\alpha'(0) = (1, 0, 1)_{(0, 0, 0)}$$

$$\|\alpha'(0)\| = (1^2 + 1^2)^{\frac{1}{2}} = \sqrt{2};$$

$$\alpha''(t) = (-2\sin t - t \cos t, 2\cos t - t \sin t, 0)_{\alpha(t)}$$

$$\alpha''(0) = (0, 2, 0)_{(0, 0, 0)}.$$

7. We first evaluate Y' and Z', differentiating each coordinate function

$$Y(t) = (\cos t, \sin t, 1)_{\alpha(t)}$$

$$Y'(t) = (-\sin t, \cos t, 0)_{\alpha(t)}$$

$$Z(t) = (\sin t, t, \cos t)_{\alpha(t)}$$

$$Z'(t) = (\cos t, 1, -\sin t)_{\alpha(t)}.$$

(a) $V(t) = Y(t) + Z(t)$

so $V'(t) = Y'(t) + Z'(t)$, by linearity,

$$= (\cos t - \sin t, \cos t + 1, -\sin t)_{\alpha(t)}.$$

(b) $V(t) = t^2\, Y(t)$

so

$V'(t) = 2t\, Y(t) + t^2 Y'(t)$, by the Leibnizian property,

$$= 2t\, (\cos t, \sin t, 1)_{\alpha(t)} + t^2(-\sin t, \cos t, 0)_{\alpha(t)}$$

$$= (2t \cos t - t^2 \sin t, 2t \sin t + t^2 \cos t, 2t)_{\alpha(t)}.$$

(c) $V(t) = Y(t)\cdot Z(t)$

so

$V'(t) = Y'(t)\cdot Z(t) + Y(t)\cdot Z'(t)$, by the Leibnizian property,

$$= (-\sin t, \cos t, 0)\cdot(\sin t, t, \cos t) + (\cos t, \sin t, 1)\cdot(\cos t, 1, -\sin t)$$

$$= -\sin^2 t + t \cos t + 0 + \cos^2 t + \sin t - \sin t$$

$$= \cos^2 t - \sin^2 t + t \cos t.$$

(d) $V(t) = Y(t) \times Z(t)$

so

$V'(t) = Y'(t) \times Z(t) + Y(t) \times Z'(t)$, by the Leibnizian property,

$$= (-\sin t, \cos t, 0)_{\alpha(t)} \times (\sin t, t, \cos t)_{\alpha(t)}$$

$$+ (\cos t, \sin t, 1)_{\alpha(t)} \times (\cos t, 1, -\sin t)_{\alpha(t)}$$

$$= (\cos^2 t, \sin t \cos t, -t \sin t - \sin t \cos t)_{\alpha(t)} +$$

$$(-1 - \sin^2 t, \cos t + \sin t \cos t, \cos t - \sin t \cos t)_{\alpha(t)}$$

$$= (\cos^2 t - \sin^2 t - 1, \cos t + 2\sin t \cos t, \cos t - t \sin t - 2\sin t \cos t)_{\alpha(t)}$$

$$= (-2\sin^2 t, \cos t + \sin 2t, \cos t - t\sin t - \sin 2t)_{\alpha(t)}.$$

8. If $\|Y\|$ has the constant value c, then

$$Y \cdot Y = c^2.$$

Differentiation gives

$$Y' \cdot Y + Y \cdot Y' = 0$$

so

$$Y \cdot Y' = 0.$$

Further differentiation gives

$$Y' \cdot Y' + Y \cdot Y'' = 0.$$

If $Y(t)$ is orthogonal to $Y''(t)$ then $(Y \cdot Y'')(t) = 0$. This forces $(Y' \cdot Y')(t) = 0$, that is $\|Y'(t)\| = 0$. The norm of a vector is zero if and only if that vector is itself zero, so this implies $Y'(t) = \mathbf{0}$, which is contrary to assumption. Thus $Y(t)$ cannot be orthogonal to $Y''(t)$.

9. Page 56, Exercise 9.

(a) $\|\alpha'\|$ is a constant

$\iff$ $\alpha' \cdot \alpha'$ is a constant

$\iff (\alpha' \cdot \alpha')' = 0$

$\iff \alpha'' \cdot \alpha' + \alpha' \cdot \alpha'' = 0$

$\iff \alpha' \cdot \alpha'' = 0$

$\iff$ α'' is always orthogonal to α'.

(b) Suppose β is the straight line $t \mapsto p + tq$, and α is the reparametrization $\beta(h)$ of β. Then

$$\alpha(u) = p + h(u)q$$

$$\alpha'(u) = h'(u)q_{\alpha(u)}$$

$$\alpha''(u) = h''(u)q_{\alpha(u)}.$$

Then $\alpha'(u)$ and $\alpha''(u)$ are both scalar multiples of the tangent vector $q_{\alpha(u)}$, and so they are collinear. Conversely, we wish to show that the collinearity of α' and α'' implies that α is a reparametrization of a straight line. By the remark about symmetry in Comment (i), it is enough to show that α has a unit-speed reparametrization $\gamma = \alpha(h)$ which is a straight line. Consequently we choose γ to be a unit-speed reparametrization of α: Theorem II.2.1 tells us that we can choose such a γ.

Now $\gamma(s) = \alpha(h(s))$

$\gamma'(s) = h'(s)\, \alpha'(h(s))$, by Lemma I.4.5,

$\gamma''(s) = h''(s)\, \alpha'(h(s)) + (h'(s))^2\, \alpha''(h(s))$, by the Leibnizian property.

$\alpha''(h(s))$ and $\alpha'(h(s))$ collinear

$\Rightarrow$ $\alpha''(h(s))$ is a scalar multiple of $\alpha'(h(s))$

$\Rightarrow$ $\gamma''(s)$ is a scalar multiple of $\alpha'(h(s))$

$\Rightarrow$ $\gamma''(s)$ and $\gamma'(s)$ are collinear.

However, γ has unit speed, that is $\|\gamma'\| = 1$, so part (a) tells us that $\gamma''(s)$ is orthogonal to $\gamma'(s)$. The only way that $\gamma''(s)$ can be both collinear and orthogonal to $\gamma'(s)$ is that $\gamma''(s) = 0$: hence by Lemma II.2.3(2) γ is a straight line.

II.3 THE FRENET FORMULAS

Introduction

This section carries straight on from Section II.2. The Frenet formulas for unit-speed curves are the first really important result in the course. You should try to become thoroughly familiar with the concepts of *tangent, principal normal, binormal, curvature* κ and *torsion* τ, which are collectively called the Frenet apparatus, and be able to find them for any unit-speed curve.

We use these ideas to define the Frenet approximation to a curve, which is a type of Taylor approximation. You have met Taylor approximations before in *Unit M100 14, Sequences and Limits II* or *Unit MST 281 10, Taylor Approximation,* and *Unit M231 13, Taylor Approximations.* The Frenet approximation is less important than the Frenet apparatus, except in so far as it helps one to visualize the shape of a curve near a point and suggests various characterizations of curves in terms of their curvature and torsion. Such a *characterization* is a statement of the form:

> β is a curve of a particular type (e.g. plane curve, cylindrical helix) if and only if the curvature (κ) and torsion (τ) of β satisfy a particular condition (e.g. $\tau = 0$; or τ/κ is constant).

The fact that there are such characterizations indicates why κ and τ are important functions associated with a curve: we shall see just how important they are in Chapter III. The section concludes with some characterizations of curves: you should note the methods of proof as much as the results. Further characterizations follow in the exercises and in the next section.

READ: Section II.3 (pages 56–63).

Comments

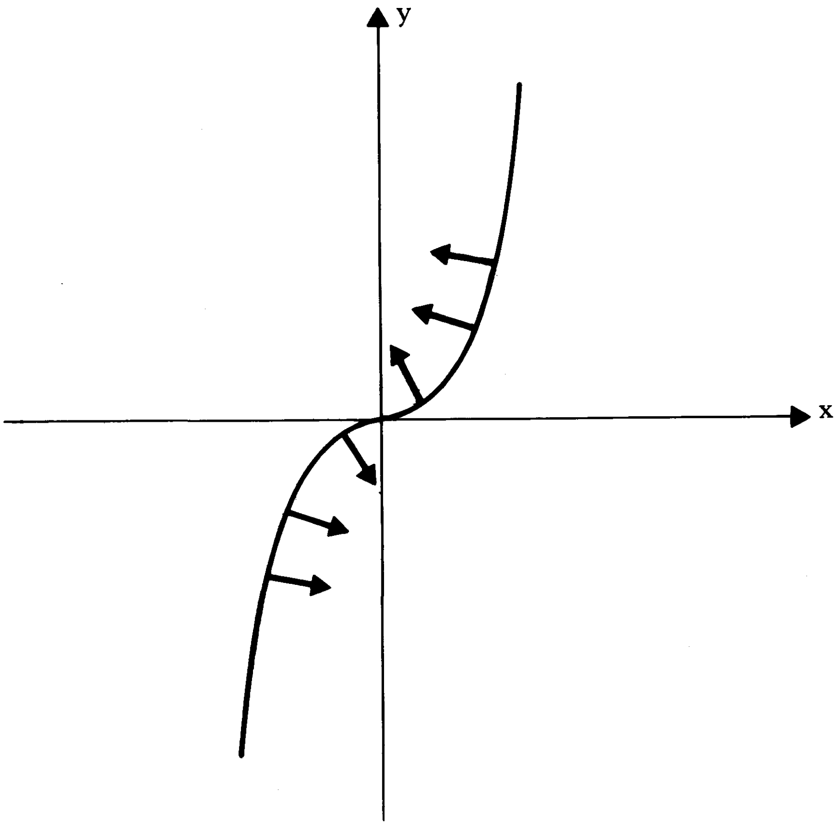
(i) Pages 57–8: curvature and principal normal The definition of κ as $\|T'\|$ forces us to avoid points where κ is zero, for at such a point not only is N undefined but κ may not be differentiable (because the square-root function is not differentiable at zero). Moreover, N may suddenly change from one side of the curve to the other near a point where κ is zero, as shown in the diagram opposite.

Consequently the footnote on page 57 is very important: we restrict our attention to portions of curves where the curvature is positive.

It is possible to develop a theory where κ is allowed to become zero and even negative, in which N is defined for many curves even at points where $\kappa = 0$. In this theory κ and N are still differentiable at such points. O'Neill does not take this approach (except for curves in E^2 – see the optional section in the exercises), but if you are interested you can read about it in T. J. Willmore: *An Introduction to Differential Geometry* (O.U.P. 1964), Chapter I.

(ii) Page 57: binormal We choose B to be T × N specifically to obtain a vector field which is orthogonal to T and N at each point. Since T(s) and N(s) are orthogonal unit vectors, Lemma II.1.A tells us that B(s) is a unit vector orthogonal to T(s) and N(s), and moreover the following equations hold:

$$\begin{aligned} T \times N &= B; & N \times T &= -B; \\ N \times B &= T; & B \times N &= -T; \\ B \times T &= N; & T \times B &= -N. \end{aligned}$$



Since T(s), N(s), B(s) form a frame at each point $\beta(s)$ we can extend the idea of orthonormal expansion by the pointwise principle to obtain the result that any vector field Y on β has an orthonormal expansion of the form

$$Y = (Y\cdot T)T + (Y\cdot N)N + (Y\cdot B)B.$$

(iii) Page 59 onwards: the correspondence between points and tangent vectors
The identification of the tangent vector v_p with the point v, for working which is concerned just with the Euclidean coordinates, enables us to omit the subscripts to tangent vectors. This convention considerably simplifies manipulation and proofs. For example, O'Neill uses this convention throughout the proof of Corollary 3.5: without this convention the last line of page 61 would read

$$\beta'(s)\cdot q_{\beta(s)} = \beta''(s)\cdot q_{\beta(s)} = 0$$

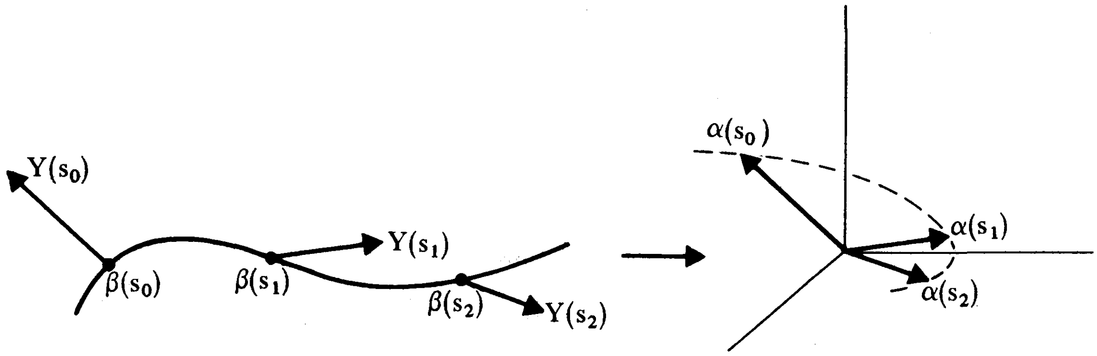
and the second paragraph on page 62 would contain words to the effect that since B is parallel there is a fixed vector v such that $B(s) = v_{\beta(s)}$ and would then define f(s) as $(\beta(s) - \beta(0))\cdot v$; this adds verbiage and symbols to the proof without enhancing our understanding of it. Consequently we use this convention whenever it is possible to do so without causing confusion. We note that by page 58 O'Neill is already writing

$$T(s) = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right)$$

rather than the technically more correct

$$T(s) = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right)_{\left(a\cos\frac{s}{c},\ a\sin\frac{s}{c},\ \frac{bs}{c}\right)}.$$

In the last paragraph on page 59 we take this identification of tangent vectors with points a little further. We associate the tangent vector $Y(s) = (y_1(s), y_2(s), y_3(s))_{\beta(s)}$ with the point $(y_1(s), y_2(s), y_3(s))$ of E^3.



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We simply take all the tangent vectors Y(s) and move the point of application $\beta(s)$ to the origin; the new end points of the tangent vectors become the new curve. If we call this new curve α, then α is parametrized in such a way that $\alpha(s_0)$ is the point with the same Euclidean coordinates as the tangent vector $Y(s_0)$. This technique allows us to transfer certain results about curves to vectors fields on curves, and vice versa.

Additional Text

The following exercise introduces a result that we shall need in the next section, so attempt the exercise now. Do not spend too long on it, and be sure to read the solution.

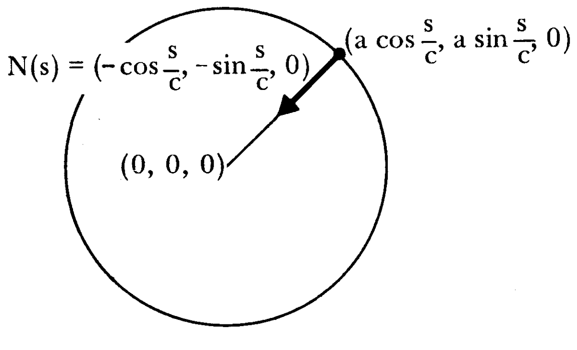
1. Page 65, Exercise 7.

Supplementary Comments

(i) Page 57: line −2 The "orthonormal expansion" referred to is that of B' in terms of T, N, B, as mentioned in Comment (ii).

$$\begin{aligned} B' &= (B'\cdot T)T + (B'\cdot N)N + (B'\cdot B)B \\ &= (B'\cdot N)N \quad \text{if } B'\cdot T = B'\cdot B = 0. \end{aligned}$$

(ii) Page 59: line 5 Looked at from above we have:



The normal is in the opposite direction to the projection of the position vector $\beta(s)$ on the xy-plane, and so it always points towards the z-axis.

(iii) Page 61: line 4 This is not the Taylor approximation to β of degree 3: that would be

$$\beta(s) \sim \beta(0) + (s - \kappa_0^2\, s^3)T_0 + \left(\kappa_0 \frac{s^2}{2} + \frac{d\kappa}{ds}(0)s^3\right)N_0 + \kappa_0 \tau_0 \frac{s^3}{6} B_0.$$

We are interested in the approximation of the component of $\beta(s) - \beta(0)$ in each of the directions T_0, N_0, B_0, so we keep just the term of lowest degree in each component.

Summary

Notation

T	Page 56, line -4
κ	Page 57, line 2
N	Page 57, line 6
B	Page 57, line 8
τ	Page 58, line 3

Definitions

(i)	Unit tangent T	Page 56, line -4
(ii)	Curvature κ	Page 57, line 2
(iii)	Principal normal N	Page 57, line 6
(iv)	Binormal B	Page 57, line 8
(v)	Frenet frame field T, N, B	Page 57, Lemma 3.1
(vi)	Torsion τ	Page 58, line 3
(vii)	Frenet approximation	Page 61, line 5
(viii)	Tangent line	Page 61, line 10
(ix)	Osculating plane	Page 61, line 14
(x)	Frenet apparatus T, N, B, κ, τ	
(xi)	Plane curve	Page 61, line -12
(xii)	Spherical curve	Page 63, last two lines of text

Example

(i) The unit-speed helix

$$\beta(s) = \left(a \cos\frac{s}{c}, a \sin\frac{s}{c}, \frac{bs}{c}\right),$$

where $c^2 = a^2 + b^2$, has constant curvature a/c^2 and constant torsion b/c^2. Page 58, Example 3.3

Results

(i) The Frenet formulas.

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= - \tau N \end{aligned}$$

Page 58, Theorem 3.2

(ii) The Frenet approximation: for small s,

$$\beta(s) \sim \beta(0) + sT(0) + \kappa(0)\frac{s^2}{2} N(0) + \kappa(0)\tau(0)\frac{s^3}{6} B(0).$$

Page 61, line 4

(iii) A unit-speed curve with positive curvature is a plane curve if and only if $\tau = 0$. Page 61, Corollary 3.5

(iv)	A unit-speed curve has zero torsion and constant curvature κ if and only if it is part of a circle with radius $1/\kappa$.	Page 59, third paragraph, and Page 62, Lemma 3.6
(v)	A spherical unit-speed curve has curvature $\geqslant 1/a$, where a is the radius of the sphere.	Page 63, last paragraph
(vi)	If $\bar{\alpha} = \alpha(h)$ and α are both unit-speed curves, then $\bar{T} = \varepsilon T(h)$, $\bar{N} = N(h)$, $\bar{B} = \varepsilon B(h)$, $\bar{\kappa} = \kappa(h)$, $\bar{\tau} = \tau(h)$, where $\varepsilon = h' = \pm 1$.	Page 65, Exercise 7
(vii)	$T = N \times B$, $N = B \times T$, $B = T \times N$.	Text, page 28
(viii)	If Y is a vector field on α, $Y = (Y\cdot T)T + (Y\cdot N)N + (Y\cdot B)B$.	Text, page 29

Techniques

(i)	Finding the Frenet apparatus T, N, B, κ, τ, by differentiation, followed by taking norms and cross products.	Page 58, Example 3.3
(ii)	Identification of a vector field on a curve with another curve.	Page 59, last paragraph
(iii)	The following two techniques are useful in characterizations:	
	(a) to show a function is constant, show that its derived function is zero, simplifying the derived function by using the Frenet formulas;	Page 62, lines 6–11
	(b) prove that "type of curve" $\Rightarrow$ "properties of κ and τ" first, as the proof should give you information which will indicate the method of proving the converse.	Page 62, proof of Lemma II.3.6

Exercises

Technique (i)

2. Page 63, Exercise 1. It is sufficient to show that the curve is part of a circle, and find its centre and radius.

Technique (ii)

3. For β as in the preceding exercise, let σ be the curve whose coordinate functions are given by $\sigma(s) = T(s)$. Find the speed of σ.

Technique (iii)

4. Page 65, Exercise 10.

Optional Section – Curves in the Plane

5. Curves in the plane are treated somewhat differently from those in E^3. If you would like to make a small sub-study of them, read the definitions given in Exercise 8 and then do that exercise. More exercises on curves in the plane will be provided later. Note that, because E^2 is two-dimensional, N spans the set of vectors orthogonal to T: T′ is orthogonal to T because $\|T\|$ = constant, and so $T' = \kappa N$ for some κ.

Solutions

1. Page 65, Exercise 7.

(a) $\bar{\alpha} = \alpha(h)$

so $\bar{\alpha}' = h'\alpha'(h)$, by Lemma I.4.5,

so $\|\bar{\alpha}'(s)\| = |h'(s)|\ \|\alpha'(h(s))\|$ for all s.

α is a unit-speed curve, so $\|\alpha'(h(s))\| = 1$ for all s; $\bar{\alpha}$ is a unit-speed curve, so $\|\bar{\alpha}'(s)\| = 1$ for all s: thus $|h'(s)| = 1$ and so $h'(s) = \pm 1$. Since h is differentiable and has an interval as its domain, h′(s) cannot be +1 for some values of s and −1 for other values of s. So we can write $h' = \varepsilon$, where $\varepsilon = \pm 1$ independent of s. Then $h(s) = \varepsilon s + h(0) = \pm s + s_0$ where $s_0 = h(0)$.

(b) The salient fact from part (a) is that $h'(s) = \varepsilon$.

$\bar{T}(s) = \alpha(h)'(s) = h'(s)\alpha'(h(s))$, by Lemma I.4.5,

$= \varepsilon T(h(s))$.

$\bar{T}'(s) = \varepsilon T(h)'(s) = \varepsilon h'(s)T'(h(s))$, by Exercise II.2.7,

$= \varepsilon^2 T'(h(s)) = T'(h(s))$:

thus

$$\bar{\kappa}(s)\,\bar{N}(s) = \kappa(h(s))\,N(h(s)): \qquad (*)$$

taking norms we find

$$|\bar{\kappa}(s)|\ \|\bar{N}(s)\| = |\kappa(h(s))|\ \|N(h(s))\|$$

so $|\bar{\kappa}(s)| = |\kappa(h(s))|$ since $\|\bar{N}(s)\| = \|N(h(s))\| = 1$:

$\kappa(h(s))$ and $\bar{\kappa}(s)$ are positive so we may remove the modulus signs, obtaining

$$\bar{\kappa}(s) = \kappa(h(s)).$$

Now this equation and (*) together imply $\bar{N}(s) = N(h(s))$ since $\kappa(h(s))$ is not zero.

$\bar{B}(s) = \bar{T}(s) \times \bar{N}(s) = \varepsilon T(h(s)) \times N(h(s)) = \varepsilon B(h(s))$.

$\bar{B}'(s) = \varepsilon B(h)'(s) = \varepsilon h'(s)\ B'(h(s))$, by Exercise II.2.7,

$= \varepsilon^2\ B'(h(s)) = B'(h(s)) = -\tau(h(s))\ N(h(s))$:

thus

$$-\bar{\tau}(s).\bar{N}(s) = -\tau(h(s))\,N(h(s)),$$

but

$$\bar{N}(s) = N(h(s))$$

so

$$\bar{\tau}(s) = \tau(h(s)).$$

The importance of this result is that if α and $\bar{\alpha}$ are two different unit-speed reparametrizations (in the sense of Theorem II.2.1) of some curve γ, then α and $\bar{\alpha}$ have the same Frenet apparatus at each point of the curve.

2. Page 63, Exercise 1.

$$\beta(s) = (\tfrac{4}{5}\cos s,\ 1 - \sin s,\ -\tfrac{3}{5}\cos s).$$

$$\beta'(s) = (-\tfrac{4}{5}\sin s,\ -\cos s,\ \tfrac{3}{5}\sin s)$$

$$\|\beta'(s)\| = (\tfrac{16}{25}\sin^2 s + \cos^2 s + \tfrac{9}{25}\sin^2 s)^{\frac{1}{2}} = (\cos^2 s + \sin^2 s)^{\frac{1}{2}} = 1,$$

so β is indeed a unit-speed curve and

$$T(s) = \beta'(s).$$

$$T'(s) = (-\tfrac{4}{5}\cos s,\ \sin s,\ \tfrac{3}{5}\cos s)$$

$$\kappa(s) = \|T'(s)\| = (\tfrac{16}{25}\cos^2 s + \sin^2 s + \tfrac{9}{25}\cos^2 s)^{\frac{1}{2}}$$
$$= (\cos^2 s + \sin^2 s)^{\frac{1}{2}} = 1, \text{ so}$$

$$N(s) = T'(s).$$

$$B(s) = T(s) \times N(s) = \begin{vmatrix} U_1 & U_2 & U_3 \\ -\tfrac{4}{5}\sin s & -\cos s & \tfrac{3}{5}\sin s \\ -\tfrac{4}{5}\cos s & \sin s & \tfrac{3}{5}\cos s \end{vmatrix}$$

$$= \left(-\tfrac{3}{5}(\cos^2 s + \sin^2 s),\ -\tfrac{12}{25}(\sin s\cos s - \sin s\cos s),\ -\tfrac{4}{5}(\sin^2 s + \cos^2 s)\right)$$

$$= \left(-\tfrac{3}{5},\ 0,\ -\tfrac{4}{5}\right).$$

$B'(s) = \mathbf{0}$ so $\tau(s) = 0$.

Since $\tau = 0$ and κ is constant, Lemma II.3.6 tells us that β is part of a circle of radius $\frac{1}{\kappa} = 1$ The proof of Lemma II.3.6 showed that the centre of the circle is given by $\beta(s) + (1/\kappa(s))\,N(s)$, which in this case gives

$$(\tfrac{4}{5}\cos s,\ 1 - \sin s,\ -\tfrac{3}{5}\cos s) + (-\tfrac{4}{5}\cos s,\ \sin s,\ \tfrac{3}{5}\cos s)$$
$$= (0, 1, 0).$$

3. σ is the curve whose coordinates are the same as those of the vector field T, and so

$$\|\sigma'(s)\| = \|T'(s)\| = \kappa(s).$$

We saw in the preceding exercise that $\kappa(s) = 1$ and so $\|\sigma'(s)\| = 1$ for all s, that is, σ has unit speed.

4. Page 65, Exercise 10.

(a) If α lies on the sphere of centre c and radius r, then

$$\|\alpha(s) - c\| = r,$$

that is,

$$(\alpha(s) - c) \cdot (\alpha(s) - c) = r^2.$$

Using coordinates only and ignoring points of application (following Comment (iii)), we differentiate to obtain

$$(\alpha(s) - c) \cdot \alpha'(s) = 0,$$

that is,

$$(\alpha(s) - c) \cdot T(s) = 0.$$

By the orthonormal expansion of $\alpha(s) - c$ in terms of the basis T(s), N(s), B(s),

$$\alpha(s) - c = a(s)N(s) + b(s)B(s)$$

where

$$a(s) = (\alpha(s) - c)\cdot N(s),\ b(s) = (\alpha(s) - c)\cdot B(s).$$

Differentiating again:

$$\alpha'(s) = a'(s)N(s) + a(s)\,N'(s) + b'(s)\,B(s) + b(s)B'(s),$$

but

$$\alpha'(s) = T(s)$$

so that

$$T(s) = a'(s)N(s) + a(s)(-\kappa(s)T(s) + \tau(s)B(s)) + b'(s)B(s) - b(s)\tau(s)N(s).$$

Since T(s), N(s), B(s), form a basis, we can equate coefficients on each side: coefficients of T(s) give

$$1 = -a(s)\kappa(s)$$

and so

$$a(s) = \frac{-1}{\kappa(s)} = -\ \rho(s);$$

coefficients of N(s) give

$$0 = a'(s) - b(s)\tau(s)$$

so

$$b(s) = \frac{1}{\tau(s)}\,a'(s)$$
$$= -\ \sigma(s)\rho'(s).$$

Thus $\alpha - c = -\ \rho N - \rho'\sigma B$.

Since N and B are orthogonal,

$$\|\alpha - c\|^2 = (-\rho)^2 + (-\rho'\sigma)^2$$

so

$$r^2 = \rho^2 + (\rho'\sigma)^2.$$

(b) We put $\gamma(s) = \alpha(s) + \rho(s)N(s) + \rho'(s)\sigma(s)B(s)$ as we know by part (a) that $\gamma(s)$ is constant when α is a spherical curve. We aim to show that $\gamma(s)$ is a constant if $\rho^2 + (\rho'\sigma)^2 = r^2$. We can achieve this if we can show that $\gamma' = 0$.

Now

$$\begin{aligned}\gamma' &= \alpha' + \rho'N + \rho N' + (\rho'\sigma)'B + \rho'\sigma B' \\ &= T + \rho'N + \rho(-\kappa T + \tau B) + (\rho'\sigma)'B - \rho'\sigma\tau N \\ &= (1 - \rho\kappa)T + (\rho' - \rho'\sigma\tau)N + (\rho\tau + (\rho'\sigma)')B \\ &= (\rho\tau + (\rho'\sigma)')B \qquad \text{since } \rho\kappa = \sigma\tau = 1.\end{aligned}$$

Now we use our information that $\rho^2 + (\rho'\sigma)^2 = r^2$. Differentiation gives

$$2\rho\rho' + 2\rho'\sigma\,(\rho'\sigma)' = 0$$

so

$$\rho + \sigma\,(\rho'\sigma)' = 0, \qquad \text{because } \rho' \neq 0,$$

so

$$\rho\tau + \sigma\tau\,(\rho'\sigma)' = 0$$

so

$$\rho\tau + (\rho'\sigma)' = 0, \qquad \text{because } \sigma\tau = 1.$$

Thus

$$\gamma' = (\rho\tau + (\rho'\sigma)')B = 0 \qquad \text{and so } \gamma \text{ is constant.}$$

Putting $\gamma(s) = c$ gives

$$\alpha(s) - c = -\rho(s)N(s) - \rho'(s)\sigma(s)B(s)$$

and so, as before,

$$\begin{aligned}\|\alpha - c\|^2 &= (-\rho)^2 + (-\rho'\sigma)^2 \\ &= \rho^2 + (\rho'\sigma)^2 \\ &= r^2 \qquad \text{by assumption}\end{aligned}$$

so

$$\|\alpha - c\| = r,$$

that is, α lies on the sphere with centre c and radius r.

5. Page 65, Exercise 8.

(a) $N = (-y', x')$, so

$N' = (-y'', x'')$.

However, $T' = (x'', y'') = \kappa N = \kappa(-y', x')$ so

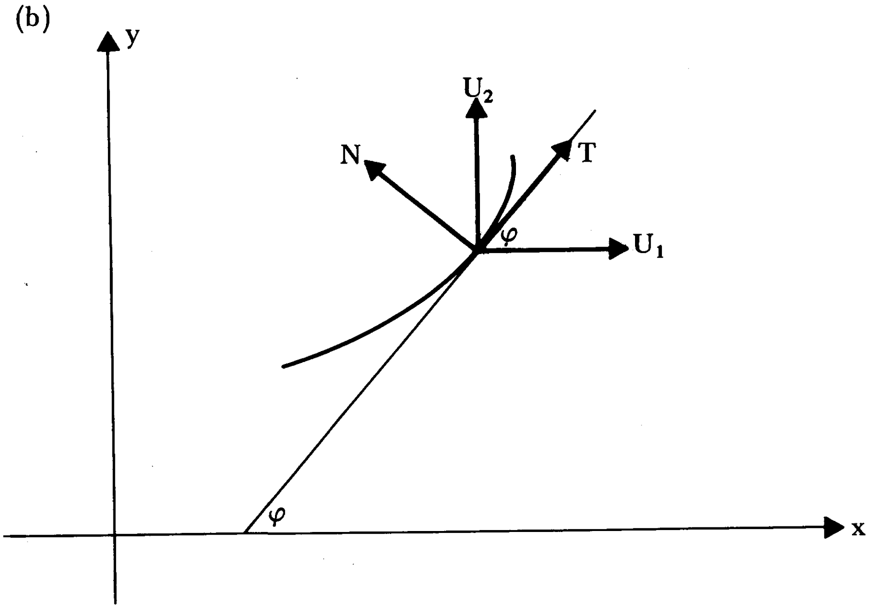
$$x'' = -\kappa y'$$

$$y'' = \kappa x'$$

and so

$$N' = (-\kappa x', -\kappa y') = -\kappa(x', y') = -\kappa T.$$

(b)



Since T has slope angle φ and unit length

$$T = \cos\varphi\, U_1 + \sin\varphi\, U_2$$

and so

$$T' = -\sin\varphi\, \varphi' U_1 + \cos\varphi\, \varphi' U_2.$$

N has slope angle $(\varphi + \frac{\pi}{2})$ and unit length, so

$$N = \cos(\varphi + \frac{\pi}{2}) U_1 + \sin(\varphi + \frac{\pi}{2}) U_2$$

$$= -\sin\varphi\, U_1 + \cos\varphi\, U_2.$$

Thus

$$T' = \varphi' N.$$

However,

$$T' = \kappa N, \text{ and so } \kappa = \varphi'.$$

II.4 ARBITRARY-SPEED CURVES

Introduction

This section extends the results of Section II.3 to arbitrary speed curves. We need to use the chain rule for reparametrizations of curves (Lemma I.4.5) and its analogue for vector fields on reparametrized curves (Exercise II.2.7) to adapt the Frenet apparatus to an arbitrary speed curve. We are searching for an effective procedure for computing the Frenet apparatus of an arbitrary regular curve, and, although we succeed in this, we lose the simplicity of the earlier Frenet formulas, with the result that the theory becomes cumbersome and the computations more involved and so liable to error. Although this whole section is important and very practical, if you find yourself becoming bogged down in calculus and computation give it a rest for a while and return later: the results of this section are not needed again until Section III.5.

READ: Section II.4 (pages 66–74), omitting the last full paragraph on page 67.

Comments

(i) Page 66: the definitions The curvature, torsion etc. of α are those at the same point of the corresponding unit-speed curve $\overline{\alpha}$, that is, if p is the point $\alpha(t) = \overline{\alpha}(s(t))$ on the curve, then the curvature at p is $\kappa(t) = \overline{\kappa}(s(t))$. There is a possible difficulty here, in that if we take different unit-speed reparametrizations of α they may give a different Frenet apparatus. However, the result of Exercise II.3.7 tells us that so long as we restrict ourselves to unit-speed reparametrizations in the sense of Theorem II.2.1 the Frenet apparatus is the same. Thus "the Frenet apparatus of α is the Frenet apparatus of any unit-speed reparametrization of α" is a valid definition.

(ii) Page 73: the orthonormal expansion of **u** By Theorem II.1.5

$$\mathbf{u} = (\mathbf{u}\cdot T)T + (\mathbf{u}\cdot N)N + (\mathbf{u}\cdot B)B$$

$$= \cos\vartheta\ T + aB \text{ for some a, because } \mathbf{u}\cdot T = \cos\vartheta \text{ and } \mathbf{u}\cdot N = 0.$$

Now **u** is a unit vector, so

$$1 = \|\mathbf{u}\|^2 = \cos^2\vartheta + a^2$$

so

$$a^2 = 1 - \cos^2\vartheta = \sin^2\vartheta$$

so

$$a = \pm\sin\vartheta.$$

If $a = +\sin\vartheta$ we have the required expression; if $a = -\sin\vartheta$ then, because $\cos(-\vartheta) = \cos\vartheta$ and $\sin(-\vartheta) = -\sin\vartheta$, we can replace ϑ by $-\vartheta$ and obtain

$$\mathbf{u} = \cos(-\vartheta)T + \sin(-\vartheta)B.$$

This is possible because we are allowing ϑ to have any value. The recognition that we can just write $\sin\vartheta$ for a in such a situation should become second nature to you.

Supplementary Comments

(i) *Page 67: second paragraph* You are not seriously expected to reparametrize the curve $\alpha(t) = (t, t^2, t^3)$. If you attempted it and got stuck then you will understand O'Neill's point — there may not be explicit formulas for $\bar{\alpha}$.

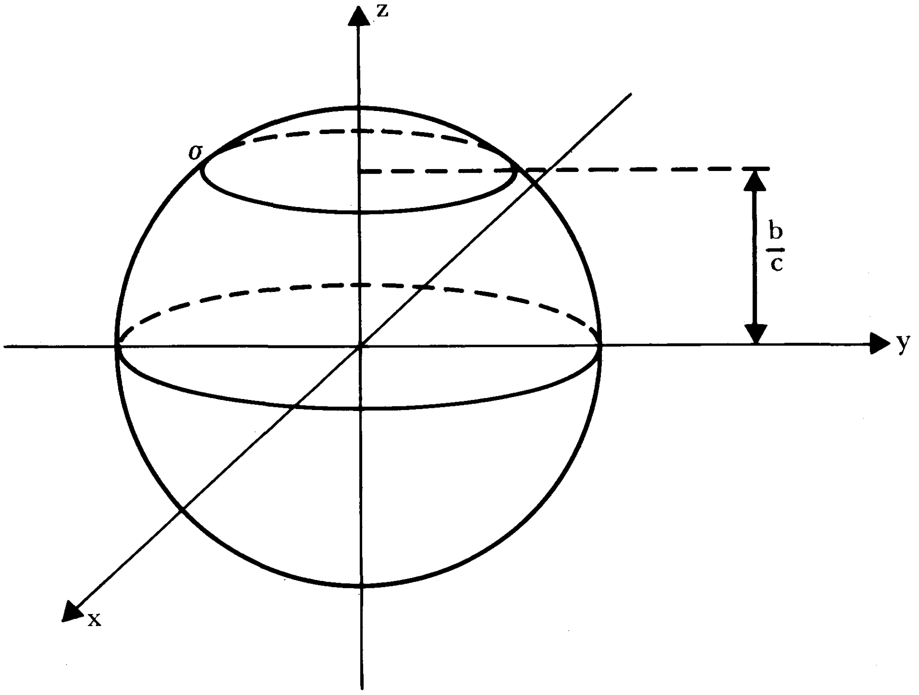
(ii) *Page 69: the formula for κ* Since $v = \|\alpha'\|$, the equation

$$\|\alpha' \times \alpha''\| = \kappa v^3$$

yields

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{v^3} = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}.$$

(iii) *Page 71: the example*



Summary

Notation

σ	Page 71, line 10

Definitions

(i)	Frenet apparatus for an arbitrary-speed curve	Page 66, first paragraph
(ii)	Spherical image σ	Page 71, line 10
(iii)	Cylindrical helix	Page 72, Definition 4.5
(iv)	Circular helix	Text: page 41, Exercise 5

Results

(i) $T' = \kappa v N$

$N' = -\kappa v T + \tau v B$

$B' = -\tau v N$ Page 67, Lemma 4.1

(ii) $\alpha' = vT$

$\alpha'' = \frac{dv}{dt}T + \kappa v^2 N$ Page 68, Lemma 4.2

(iii) $T = \frac{\alpha'}{\|\alpha'\|}$ $\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}$

$B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}$ $\tau = \frac{(\alpha' \times \alpha'')\cdot\alpha'''}{\|\alpha' \times \alpha''\|^2}$

$N = B \times T$ Page 69, Theorem 4.3

(iv) For a regular curve:

$\kappa = 0$	if and only if α is a straight line
$\tau = 0$	if and only if α is planar
κ constant >0, $\tau = 0$	if and only if α is part of a circle
κ constant >0, τ constant $\neq 0$	if and only if α is a circular helix
τ/κ constant	if and only if α is a cylindrical helix.

Page 74

Techniques

(i)	Finding the Frenet apparatus of an arbitrary regular curve.	Page 69, Theorem 4.3
(ii)	Finding the spherical image of a curve.	Page 71, second paragraph
(iii)	Finding the Frenet apparatus for a curve derived from some other curve in terms of the latter's Frenet apparatus: this involves repeated application of the Frenet formulas and taking higher derivatives.	Page 71, last paragraph
(iv)	Finding **u** and ϑ for a cylindrical helix by putting $\cot\vartheta = \tau/\kappa$ and $\mathbf{u} = \cos\vartheta\, T + \sin\vartheta\, B$.	Page 73, Theorem 4.6

Exercises

Technique (i)

1. Page 74, Exercise 2. This continues the example of Exercise II.2.3, whose solution is on Text page 23.

Techniques (ii) and (iii)

2. Page 75, Exercise 11. κ_σ has been found on page 72, so all you need find is τ_σ. Make use of the working already done on pages 71–72.

Technique (iii)

3. Page 74, Exercise 8.

Technique (iv)

4. Page 75, Exercise 9(c).

Theory Exercises (omit if you are short of time)

5. Page 75, Exercise 10. This continues from Exercise II.4.8: a cylindrical helix β is a circular helix if its cross-section curve γ is part of a circle.

6. Page 75, Exercise 12(a). This continues from Exercise II.4.11.

Optional Section — Curves in the Plane

If you have time to follow this optional line of study, you should also do:

7. Let α be a plane curve, $\bar{\alpha}$ a unit-speed reparametrization of α, that is $\alpha = \bar{\alpha}(s)$ where s is an arc-length function for α. Define

$$T = \bar{T}(s)$$

$$\kappa = \bar{\kappa}(s)$$

N as in Exercise 8 on page 65.

Prove that (a) $\alpha' = vT$

(b) $T' = v\kappa N$

(c) $N' = -\kappa v T$.

8. Page 75, Exercise 14.

9. Page 75, Exercise 15. The central curve is defined in Exercise II.4.13. After you have sketched both curves, how do you account for the cusp (sharp point) in α^* at $\alpha^*(0)$?

Solutions

1. Page 74, Exercise 2.

$\alpha(t) = (\cosh t, \sinh t, t)$.

We already know that:

$\alpha'(t) = (\sinh t, \cosh t, 1)$

$v(t) = \sqrt{2} \cosh t$

$s(t) = \sqrt{2} \sinh t$.

Now

$\alpha''(t) = (\cosh t, \sinh t, 0)$

$\alpha'''(t) = (\sinh t, \cosh t, 0)$

so $\alpha'(t) \times \alpha''(t) = (-\sinh t, \cosh t, \sinh^2 t - \cosh^2 t) = (-\sinh t, \cosh t, -1)$,

$\|\alpha'(t) \times \alpha''(t)\| = (\sinh^2 t + \cosh^2 t + 1)^{\frac{1}{2}} = \sqrt{2} \cosh t$,

$\alpha'(t) \times \alpha''(t) \cdot \alpha'''(t) = -\sinh^2 t + \cosh^2 t = 1$.

Now we simply apply Theorem II.4.3:

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|} = \frac{1}{\sqrt{2}}(\tanh t, 1, \operatorname{sech} t) \text{ (writing tanh for } \frac{\sinh}{\cosh} \text{ and sech for } \frac{1}{\cosh}\text{)}$$

$$B(t) = \frac{\alpha'(t) \times \alpha''(t)}{\|\alpha'(t) \times \alpha''(t)\|} = \frac{1}{\sqrt{2}}(-\tanh t, 1, -\operatorname{sech} t)$$

$$N(t) = B(t) \times T(t) = \frac{1}{2}(2\operatorname{sech} t, 0, -2\tanh t) = (\operatorname{sech} t, 0, -\tanh t)$$

$$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3} = \frac{\sqrt{2}\cosh t}{(\sqrt{2}\cosh t)^3} = \frac{1}{2\cosh^2 t}$$

$$\tau(t) = \frac{\alpha'(t) \times \alpha''(t) \cdot \alpha'''(t)}{\|\alpha'(t) \times \alpha''(t)\|^2} = \frac{1}{(\sqrt{2}\cosh t)^2} = \frac{1}{2\cosh^2 t}$$

Since we know $s(t) = \sqrt{2}\sinh t$, $2\cosh^2 t = 2 + 2\sinh^2 t = 2 + s^2$,

so $\kappa(s) = \tau(s) = \dfrac{1}{2 + s^2}$.

2. Page 75, Exercise 11.

Let β have Frenet apparatus T, N, B, κ, τ.

Making the usual identification of coordinates and ignoring both the distinction between points and tangent vectors and the points of application of tangent vectors, we can write, as on pages 71–72:

$$\sigma = T$$

$$\sigma' = \kappa N$$

$$\sigma'' = -\kappa^2 T + \kappa' N + \kappa\tau B$$

$$\sigma' \times \sigma'' = \kappa^2(\kappa B + \tau T)$$

$$\|\sigma' \times \sigma''\| = \kappa^2(\kappa^2 + \tau^2)^{\frac{1}{2}}.$$

Now

$$\begin{aligned} \sigma''' &= -2\kappa\kappa' T - \kappa^2 T' + \kappa'' N + \kappa' N' + \kappa'\tau B + \kappa\tau' B + \kappa\tau B' \\ &= -2\kappa\kappa' T - \kappa^3 N + \kappa'' N - \kappa'\kappa T + \kappa'\tau B + \kappa'\tau B + \kappa\tau' B - \kappa\tau^2 N \\ &= -3\kappa\kappa' T + (\kappa'' - \kappa^3 - \kappa\tau^2)N + (\kappa\tau' + 2\kappa'\tau)B \end{aligned}$$

so

$$\begin{aligned} \sigma' \times \sigma'' \cdot \sigma''' &= \kappa^2\tau(-3\kappa\kappa') + \kappa^3(\kappa\tau' + 2\kappa'\tau) \\ &= \kappa^3(\kappa\tau' - \tau\kappa') \\ &= \kappa^5(\tau/\kappa)'. \end{aligned}$$

$$\tau_\sigma = \frac{\sigma' \times \sigma'' \cdot \sigma'''}{\|\sigma' \times \sigma''\|^2} = \frac{\kappa^5(\tau/\kappa)'}{\kappa^4(\kappa^2 + \tau^2)} = \frac{(\tau/\kappa)'}{\kappa(1 + (\tau/\kappa)^2)}$$

3. Page 74, Exercise 8.

(a) We have to show that $f(t)' = (\gamma(t) - \alpha(0)) \cdot \mathbf{u} = 0$.

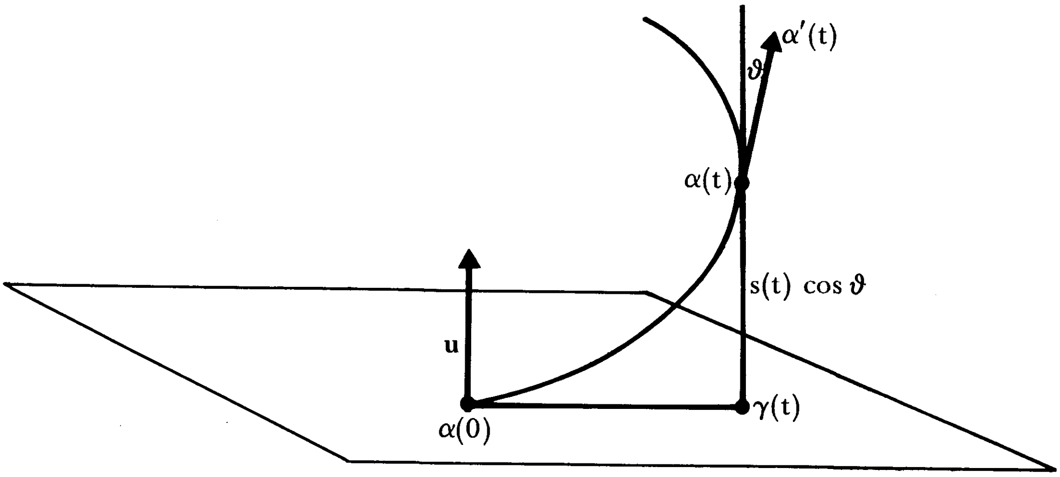
We know that

$$f(0) = (\gamma(0) - \alpha(0)) \cdot \mathbf{u} = s(0)\cos\vartheta\; \mathbf{u} \cdot \mathbf{u} = 0$$

because $s(0) = 0$, and so it is sufficient to show that f is a constant function, i.e. that $f' = 0$.

$$\begin{aligned} f(t) &= (\gamma(t) - \alpha(0))\cdot\mathbf{u} \\ &= (\alpha(t) - s(t)\cos\vartheta\,\mathbf{u} - \alpha(0))\cdot\mathbf{u} \\ &= (\alpha(t) - \alpha(0))\cdot\mathbf{u} - s(t)\cos\vartheta \qquad \text{because } \mathbf{u}\cdot\mathbf{u} = 1; \\ f'(t) &= \alpha'(t)\cdot\mathbf{u} - s'(t)\cos\vartheta \\ &= v(t)\,T(t)\cdot\mathbf{u} - v(t)\cos\vartheta \\ &= v(t)\cos\vartheta - v(t)\cos\vartheta \\ &= 0. \end{aligned}$$

Hence $\gamma(t) - \alpha(0)$ is always perpendicular to **u**, that is, γ lies in the plane through $\alpha(0)$ orthogonal to **u**.



(b) Let us suppose that α has unit-speed, so that $s(t) = t$ and $v(t) = 1$. Using bars to denote the Frenet apparatus of γ:

$$\begin{aligned} \gamma(t) &= \alpha(t) - t\cos\vartheta\,\mathbf{u} \\ \gamma'(t) &= \alpha'(t) - \cos\vartheta\,\mathbf{u} = T(t) - \cos\vartheta\,\mathbf{u}. \qquad (*) \end{aligned}$$

$$\begin{aligned} \bar{v} &= \|\gamma'\| \\ &= ((T - \cos\vartheta\,\mathbf{u})\cdot(T - \cos\vartheta\,\mathbf{u}))^{\frac{1}{2}} \\ &= (T\cdot T - 2\cos\vartheta\,T\cdot\mathbf{u} + \cos^2\vartheta\,\mathbf{u}\cdot\mathbf{u})^{\frac{1}{2}} \\ &= (1 - 2\cos^2\vartheta + \cos^2\vartheta)^{\frac{1}{2}} \\ &= (1 - \cos^2\vartheta)^{\frac{1}{2}} \\ &= (\sin^2\vartheta)^{\frac{1}{2}} = |\sin\vartheta|. \end{aligned}$$

Since $\bar{v}$ is constant, Lemma II.4.2 gives:

$$\begin{aligned} \gamma''(t) &= \bar{\kappa}(t)\,(\bar{v}(t))^2\,\bar{N}(t) \\ &= \bar{\kappa}(t)\sin^2\vartheta\,\bar{N}(t), \end{aligned}$$

but our first expression, (*), for $\gamma'(t)$ gives:

$$\gamma''(t) = T'(t) = \kappa(t)N(t),$$

so $\bar{\kappa}(t) \sin^2 \vartheta\ \bar{N}(t) = \kappa(t)N(t)$.

Taking norms gives $\bar{\kappa}(t) = (\kappa(t)/\sin^2 \vartheta)$ since $\kappa(t), \bar{\kappa}(t)$, and $\sin^2 \vartheta$ are all positive.

4. Page 75, Exercise 9(c).

We use the technique used in the proof of Theorem II.4.6. We already have all the necessary information about α (see solution to Exercise II.4.2).

$\tau(t) = \kappa(t) = \dfrac{1}{2 \cosh^2 t}$, so $\tau(t)/\kappa(t) = 1$: we choose

$$\vartheta = \frac{\pi}{4} \text{ so that } \cot\vartheta = 1 = \frac{\tau(t)}{\kappa(t)}.$$

Now we put $U(t) = \cos\frac{\pi}{4}\, T(t) + \sin\frac{\pi}{4}\, B(t)$

$$= \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\ (\tanh t,\ 1,\ \operatorname{sech} t) + \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\ (-\tanh t,\ 1,\ -\operatorname{sech} t)$$

$$= (0, 1, 0).$$

This vector field U is parallel, as expected, so we put $\mathbf{u} = (0, 1, 0)$, and $\mathbf{u}$ is a fixed unit vector such that $T(t)\cdot\mathbf{u} = \cos(\pi/4)$.

$$\begin{aligned}\gamma(t) &= \alpha(t) - s(t) \cos\frac{\pi}{4}\, \mathbf{u} \\ &= (\cosh t,\ \sinh t,\ t) - \sqrt{2} \sinh t.\ \frac{1}{\sqrt{2}}\ (0, 1, 0) \\ &= (\cosh t,\ 0,\ t).\end{aligned}$$

$$\gamma(t) - \alpha(0) = (\cosh t,\ 0,\ t) - (1, 0, 0) = (\cosh t - 1, 0, t)$$

so $(\gamma(t) - \alpha(0))\cdot\mathbf{u} = (\cosh t - 1, 0, t)\cdot(0, 1, 0) = 0$, as required.

5. Page 75, Exercise 10.

Since $\kappa > 0$ and $\tau \neq 0$ are constants, we can apply Theorem II.4.6: thus β is certainly a cylindrical helix with

$$\vartheta = \cot^{-1}(\tau/\kappa), \qquad \mathbf{u} = \cos\vartheta\ T(t) + \sin\vartheta\ B(t).$$

All that remains to show is that the cross-section curve γ of β is part of a circle. By Exercise II.4.8(a), γ is planar; by Exercise II.4.8(b) the curvature of γ is $(\kappa/\sin^2\vartheta)$, which is a constant: hence, by Lemma II.3.6 (which is unaltered if the unit-speed restriction is lifted), γ is part of a circle.

6. Page 75, Exercise 12(a).

We may suppose the curve β has unit-speed. Let σ be its spherical image.

β is a cylindrical helix $\Longleftrightarrow$ τ/κ is constant (by Theorem II.4.6)

$\Longleftrightarrow$ κ_σ is constant and τ_σ is zero (by Exercise II.4.11)

$\Longleftrightarrow$ σ is part of a circle (by Lemma II.3.6 and its converse).

7. (a) $\alpha = \bar{\alpha}(s)$

so

$\alpha' = s'\bar{\alpha}'(s)$, by Lemma I.4.5,

$= v\bar{\alpha}'(s)$, because $s' = v$,

$= v\bar{T}(s)$

$= vT.$

(b) $T = \bar{T}(s)$

so

$T' = s'\bar{T}'(s)$, by Exercise II.2.7,

$= v\bar{T}'(s)$

$= v\bar{\kappa}(s)\bar{N}(s)$, because $\bar{T}' = \bar{\kappa}\bar{N}$,

$= v\kappa N.$

(c) $N = \bar{N}(s)$

so

$N' = s'\bar{N}'(s)$, by Exercise II.2.7,

$= v\bar{N}'(s)$

$= v(-\bar{\kappa}(s)\bar{T}(s))$, by Exercise II.3.8,

$= -v\kappa T.$

8. Page 75, Exercise 14.

$\alpha' = vT$

$\alpha'' = v'T + vT'$

$= v'T + v^2\kappa N$

Thus $\alpha''\cdot N = v^2\kappa$.

However, in the plane $N = J(T) = \dfrac{J(\alpha')}{v}$, so

$$\alpha'' \cdot \frac{J(\alpha')}{v} = v^2\kappa$$

i.e.

$$\kappa = \frac{\alpha''\cdot J(\alpha')}{v^3}.$$

In terms of the coordinate functions,

$\alpha' = (x', y')_\alpha$

$J(\alpha') = (-y', x')_\alpha$

$v = (x'^2 + y'^2)^{\frac{1}{2}}$

$\alpha'' = (x'', y'')_\alpha$

so $$\kappa = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{\frac{3}{2}}}.$$

9. Page 75, Exercise 15.

(a) $\alpha^* = \alpha + \frac{1}{\kappa} N$

$$\alpha^{*\prime} = \alpha' + \frac{1}{\kappa} N' - \frac{\kappa'}{\kappa^2} N$$

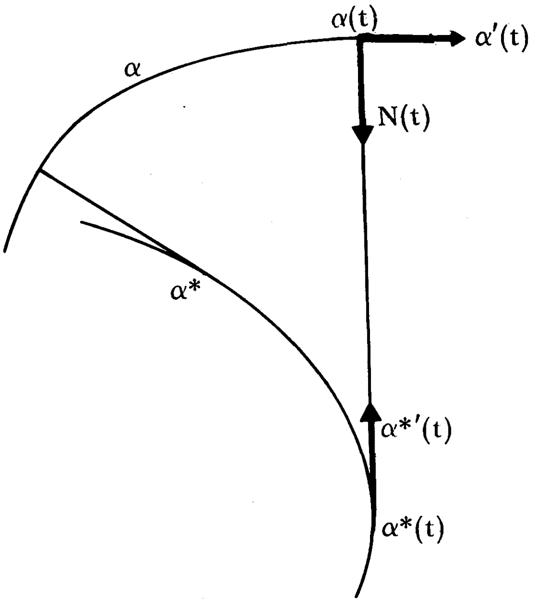
$$= vT - \frac{v\kappa}{\kappa} T - \frac{\kappa'}{\kappa^2} N$$

$$= -\frac{\kappa'}{\kappa^2} N.$$

The tangent line to α^* at $\alpha^*(t)$ is the line through $\alpha^*(t)$ in the direction $\alpha^{*\prime}(t)$; the normal line to α at $\alpha(t)$ is the line through $\alpha(t)$ in the direction $N(t)$, that is in the direction orthogonal to $\alpha'(t)$. Consequently these two lines are the same if and only if

(i) $\alpha^{*\prime}(t)$ and $\alpha'(t)$ are orthongonal: $\alpha^{*\prime}(t) \cdot \alpha'(t) = 0$

(ii) there is a number $u(t)$ such that $\alpha^*(t) = \alpha(t) + u(t)N(t)$.



Firstly, let us check that α^* does satisfy these conditions.

$$\alpha^{*\prime}(t) \cdot \alpha'(t) = \frac{-\kappa'(t)}{(\kappa(t))^2} N(t) \cdot \alpha'(t) = 0$$

so (i) is satisfied. Moreover

$$\alpha^*(t) = \alpha(t) + \frac{1}{\kappa(t)} N(t)$$

so condition (ii) is satisfied with $u(t) = \frac{1}{\kappa(t)}$.

Conversely, suppose β is a curve satisfying (i) and (ii), that is

$$\beta' \cdot \alpha' = 0$$

and

$$\beta = \alpha + uN.$$

Then

$$\begin{aligned}\beta' &= \alpha' + u'N + uN' \\ &= vT + u'N - uv\kappa T \\ &= v(1 - u\kappa)T + u'N\end{aligned}$$

so

$$\begin{aligned}\beta'\cdot\alpha' &= (v(1 - u\kappa)T + u'N)\cdot(vT) \\ &= v^2(1 - u\kappa).\end{aligned}$$

$\beta'\cdot\alpha' = 0$ and v is non-zero, so this implies $\kappa = \frac{1}{u}$, that is

$$\beta = \alpha + \frac{1}{\kappa}N = \alpha^*.$$

(b) We have already shown that $N = \frac{J(\alpha')}{v}$, $\kappa = \frac{\alpha''\cdot J(\alpha')}{v^3}$,

$$\begin{aligned}\text{so } \alpha^* &= \alpha + \frac{1}{\kappa}N \\ &= \alpha + \frac{v^3}{\alpha''\cdot J(\alpha')}\,\frac{J(\alpha')}{v} \\ &= \alpha + \frac{v^2}{\alpha''\cdot J(\alpha')}J(\alpha') \\ &= \alpha + \frac{\alpha'\cdot\alpha'}{\alpha''\cdot J(\alpha')}J(\alpha').\end{aligned}$$

(c) $\alpha(t) = (t + \sin t,\ 1 + \cos t) \qquad -\pi < t < \pi$

$\alpha'(t) = (1 + \cos t,\ -\sin t)$

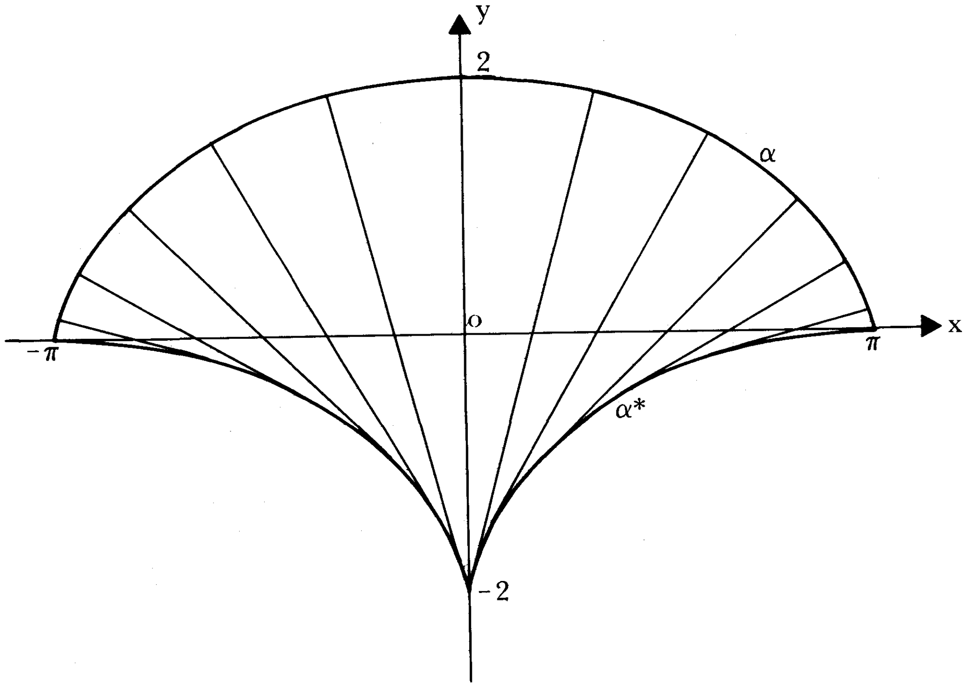
$\alpha''(t) = (-\sin t,\ -\cos t)$

$\alpha'(t)\cdot\alpha'(t) = 1 + 2\cos t + \cos^2 t + \sin^2 t = 2 + 2\cos t$

$J(\alpha'(t)) = (\sin t,\ 1 + \cos t)$

$\alpha''(t)\cdot J(\alpha'(t)) = -\sin^2 t - \cos t - \cos^2 t = -1 - \cos t.$

$$\begin{aligned}\text{Now } \alpha^*(t) &= \alpha(t) + \frac{\alpha'(t)\cdot\alpha'(t)}{\alpha''(t)\cdot J(\alpha'(t))}J(\alpha'(t)) \\ &= \alpha(t) - 2J(\alpha'(t)) \\ &= (t + \sin t,\ 1 + \cos t) - 2(\sin t,\ 1 + \cos t) \\ &= (t - \sin t,\ -1 - \cos t).\end{aligned}$$



The cusp at $\alpha^*(0)$ suggests that α^* is not regular at that point. In fact that is the case, for

$$\alpha^{*\prime}(t) = (1 - \cos t, \sin t)$$

so

$$\alpha^{*\prime}(0) = (0, 0).$$

II.5 COVARIANT DERIVATIVES

Introduction

This section depends on Sections I.3 and II.2; you also need to be familiar with the differential of a function from E^3 to R (Section I.5). We introduce a new derivative, the *covariant derivative*. This is a very important idea in differential geometry, and serves as the starting point for a lot of more advanced work. The definition is similar to the other definitions of derivatives in the book, in that it is given in terms of a derivative we already know along the particular curve $t \longmapsto p + tv$. It generalizes the derivative of a vector field on a curve.

READ: Section II. 5 (pages 77-80).

Comment

(i) *Page 78: the symbol* ∇ This is usually pronounced "del" or "nabla".

Additional Text

(i) *Covariant Differential* If $W = \Sigma w_i U_i$ is a vector field on E^3, the *covariant differential*, ∇W, of W is the function on tangent vectors defined by

$$\nabla W(v) = \nabla_v W.$$

By Lemma II.5.2 $\nabla_v W = \Sigma v[w_i]\, U_i(p)$ and so

$$\begin{aligned}\nabla W(v) &= \Sigma v[w_i]\, U_i(p)\\ &= \Sigma dw_i(v) U_i(p), \qquad \text{by definition of } dw_i,\\ &= (\Sigma dw_i U_i)(v).\end{aligned}$$

Thus we may write

$$\nabla W = \Sigma dw_i U_i.$$

(ii) *A Useful Result* The result of the following exercise will be used often later in the course. Attempt the exercise now.

1. Page 81, Exercise 6(a). (HINT: Use Lemma I.4.6.)

Summary

Notation

$\nabla_v W$	Page 78, line 2
$\nabla_V W$	Page 79, line 15
∇W	Text, page 49

Definitions

(i)	Covariant derivative with respect to a tangent vector $\nabla_v W$	Page 77, Definition 5.1
(ii)	Covariant derivative with respect to a vector field $\nabla_V W$	Page 79, line 12
(iii)	Covariant differential ∇W	Text, page 49

Results

(i)	If $W = \Sigma w_i U_i$, $\nabla_v W = \Sigma v[w_i]\, U_i(p)$	Page 78, Lemma 5.2
	$\nabla_V W = \Sigma V[w_i]\, U_i$.	Page 79, line 18
(ii)	Linearity and Leibnizian properties of the covariant derivative.	
	With respect to a tangent vector v:	
	$\nabla_{av+bw} Y = a\nabla_v Y + b\nabla_w Y$	
	$\nabla_v(aY + bZ) = a\nabla_v Y + b\nabla_v Z$	
	$\nabla_v(fY) = v[f]\, Y(p) + f(p)\nabla_v Y$	
	$v[Y \cdot Z] = \nabla_v Y \cdot Z(p) + Y(p) \cdot \nabla_v Z$	Page 78, Theorem 5.3
	With respect to a vector field V:	
	$\nabla_{fV+gW} Y = f\nabla_V Y + g\nabla_W Y$	
	$\nabla_V(aY + bZ) = a\nabla_V Y + b\nabla_V Z$	
	$\nabla_V(fY) = V[f]\, Y + f\nabla_V Y$	
	$V[Y \cdot Z] = \nabla_V Y \cdot Z + Y \cdot \nabla_V Z$	Page 80, Corollary 5.4
(iii)	If $W = \Sigma w_i U_i$, $\nabla W = \Sigma\, dw_i U_i$.	Text, page 49
(iv)	$\nabla_{\alpha'(t)} W = (W(\alpha))'(t)$.	Page 81, Exercise 6(a)

Techniques

(i)	Finding the covariant derivative with respect to a tangent vector or vector field,		
	(a)	from first principles;	Page 77, Definition 5.1
	(b)	using result (i) and calculating the directional derivatives from first principles;	Page 78, Lemma 5.2
	(c)	using result (i), the linearity and Leibnizian properties of both the directional and covariant derivative, and the results $U_i[f] = (\partial f/\partial x_i)$, $v[f] = \Sigma\, v_i(\partial f/\partial x_i)$;	Page 78, Lemma 5.2, Page 78, Theorem 5.3 and Page 80, Corollary 5.4
	(d)	by means of the covariant differential.	Text, page 49

Exercises

Technique (i)(a)

2. Page 80, Exercise 1(a). Work directly from Definition II.5.1.

Technique (i)(b)

3. Page 80, Exercise 1(a). Use the result of Lemma II.5.2 but calculate the directional derivatives from first principles.

Technique (i)(c)

4. Page 80, Exercise 1(a). Now use all the results available to simplify the work!

5. Page 80, Exercise 2(a).

6. Page 80, Exercise 2(c).

Technique (i)(d)

7. Page 80, Exercise 5.

Solutions

1. Page 81, Exercise 6(a).

Let $W = \Sigma\, w_i U_i$.

Then $\nabla_{\alpha'(t)}W = \sum_i \alpha'(t)[w_i]\, U_i(\alpha(t))$, by Lemma II.5.2,

$= \sum_i \dfrac{d(w_i(\alpha))}{dt}(t) U_i(\alpha(t))$, by Lemma I.4.6,

$= (\sum_i w_i(\alpha(t)) U_i(\alpha(t)))'$, by the definition of differentiation of a vector field on a curve,

$= (W_\alpha)'(t)$.

2. Page 80, Exercise 1(a).

$p = (1, 3, -1), v = (1, -1, 2)$, so

$p + tv = (1 + t, 3 - t, -1 + 2t)$.

$W = x^2 U_1 + yU_2$

so $W(p + tv) = (1 + t)^2 U_1(p + tv) + (3 - t)U_2(p + tv)$.

$\nabla_v W = W(p + tv)'(0)$

$= [2(1 + t)U_1(p + tv) - U_2(p + tv)]\big|_{t = 0}$

$= 2U_1(p) - U_2(p)$

$= (2, -1, 0)_p$.

3. Page 80, Exercise 1(a).

$$W = x^2U_1 + yU_2$$

so

$$\nabla_v W = v[x^2]\,U_1(p) + v[y]\,U_2(p) \qquad \text{by Lemma II.5.2.}$$

$$\begin{aligned} v[x^2] &= \frac{d}{dt}x^2(p+tv)\Big|_{t=0} \\ &= \frac{d}{dt}(1+t)^2\Big|_{t=0} \\ &= 2(1+t)\Big|_{t=0} \\ &= 2; \\ v[y] &= \frac{d}{dt}y(p+tv)\Big|_{t=0} \\ &= \frac{d}{dt}(3-t)\Big|_{t=0} \\ &= -1\Big|_{t=0} \\ &= -1. \end{aligned}$$

Thus

$$\begin{aligned} \nabla_v W &= 2U_1(p) - U_2(p) \\ &= (2,-1,0)_p. \end{aligned}$$

4. Page 80, Exercise 1(a).

$$\nabla_v W = v[x^2]\,U_1(p) + v[y]\,U_2(p).$$

$$\begin{aligned} v[x^2] &= v_1\frac{\partial(x^2)}{\partial x}(p) + v_2\frac{\partial(x^2)}{\partial y}(p) + v_3\frac{\partial(x^2)}{\partial z}(p) \\ &= v_1\, 2x(p) \\ &= 1.2.1 = 2; \\ v[y] &= v_1\frac{\partial y}{\partial x}(p) + v_2\frac{\partial y}{\partial y}(p) + v_3\frac{\partial y}{\partial z}(p) \\ &= v_2 \\ &= -1, \end{aligned}$$

so

$$\begin{aligned} \nabla_v W &= 2U_1(p) - U_2(p) \\ &= (2,-1,0)_p. \end{aligned}$$

5. Page 80, Exercise 2(a).

$$W = \cos x \; U_1 + \sin x \; U_2$$

so

$$\begin{aligned}
\nabla_V W &= V[\cos x]\, U_1 + V[\sin x]\, U_2 && \text{by Lemma II.5.2.} \\
V[\cos x] &= (-yU_1 + xU_3)[\cos x] \\
&= -y\, U_1[\cos x] + xU_3[\cos x], && \text{by linearity,} \\
&= -y \frac{\partial(\cos x)}{\partial x} + x \frac{\partial(\cos x)}{\partial z} \\
&= y \sin x + 0; \\
V[\sin x] &= (-yU_1 + xU_3)[\sin x] \\
&= -yU_1[\sin x] + xU_3[\sin x], && \text{by linearity,} \\
&= -y \frac{\partial(\sin x)}{\partial x} + x \frac{\partial(\sin x)}{\partial z} \\
&= -y \cos x + 0.
\end{aligned}$$

Thus

$$\nabla_V W = y \sin x \; U_1 - y \cos x \; U_2.$$

6. Page 80, Exercise 2(c).

$$\begin{aligned}
\nabla_V(z^2W) &= V[z^2]\,W + z^2\nabla_V W && \text{by Corollary II.5.4.} \\
V[z^2] &= 2zV[z], && \text{by Corollary I.3.4,} \\
&= 2z(-yU_1 + xU_3)[z] \\
&= -2yz\, U_1[z] + 2xz\, U_3[z], && \text{by linearity,} \\
&= -2yz.0 + 2xz.1 \\
&= 2xz,
\end{aligned}$$

so

$$\begin{aligned}
\nabla_V(z^2W) &= 2xz\, W + z^2\nabla_V W \\
&= 2xz\,(\cos x\; U_1 + \sin x\; U_2) + z^2(y \sin x\; U_1 - y \cos x\; U_2) \\
&= (2xz \cos x + yz^2 \sin x)U_1 + (2xz \sin x - yz^2 \cos x)U_2.
\end{aligned}$$

7. Page 80, Exercise 5.

$$W = xy^3U_1 - x^2z^2U_3$$

so

$$\begin{aligned}
\nabla W &= d(xy^3)U_1 - d(x^2z^2)U_3 \\
&= (y^3dx + 3xy^2dy)U_1 - (2xz^2dx + 2x^2z\, dz)U_3.
\end{aligned}$$

(a) $(y^3dx + 3xy^2dy)\,((1, 0, -3)_{(-1,\, 2,\, -1)}) = 2^3.1 + 3.(-1).2^2.0 = 8$

$(2xz^2\, dx + 2x^2 z dz)\,((1, 0, -3)_{(-1,\, 2,\, -1)}) = 2.(-1).(-1)^2.1 + 2.(-1)^2.(-1).(-3) = 4$

and so

$$\begin{aligned}
\nabla W\,((1, 0, -3)_{(-1,\, 2,\, -1)}) &= 8U_1(p) - 4U_3(p) \\
&= (8, 0, -4)_p.
\end{aligned}$$

(b) $(y^3dx + 3xy^2dy)\,((-1, 2, -1)_{(1, 3, 2)}) = 3^3.(-1) + 3.1.3^2.2 = 27$

$(2xz^2dx + 2x^2z\,dz)\,((-1, 2, -1)_{(1, 3, 2)}) = 2.1.2^2(-1) + 2.1^2.2.(-1) = -12$

and so

$$\begin{aligned}\nabla W((-1, 2, -1)_{(1, 3, 2)}) &= 27U_1(p) + 12U_3(p)\\ &= (27, 0, 12)_p.\end{aligned}$$

II.6 FRAME FIELDS

Introduction

This section does not depend on the previous two, but on Sections II.1 and II.3. The idea of a general frame field (as opposed to the Frenet frame field of Section II.3) is formulated.

READ: Section II.6 (pages 81-84).

Additional Text

(i) *The Pointwise Principle* In this section O'Neill has used the pointwise principle to extend some familiar definitions and theorems from the set of all tangent vectors to E^3 to the set S of all vector fields V in E^3. We recall that S is only a three-dimensional *module* (see Comment (ii) to Section I.2 (Text, page 10)) and so S is not isomorphic to E^3. However, because each element of S can be expressed as $\Sigma f_i U_i$, many of the ideas do carry over to S via the pointwise principle with no trouble. Some examples are given below.

(a) The vector fields E_1, E_2, E_3 form a basis for S if the tangent vectors $E_1(p)$, $E_2(p)$, $E_3(p)$ form a basis for $T_p(E^3)$ for each point p of E^3.

(b) The dot and cross products and norm are defined pointwise. That is, if V and W are vector fields then V·W is the function from E^3 to R defined by

$$(V \cdot W)(p) = V(p) \cdot W(p),$$

V × W is the vector field defined by

$$(V \times W)(p) = V(p) \times W(p),$$

and $\|V\|$ is the function from E^3 to R defined by

$$\|V\|(p) = \|V(p)\|.$$

(c) Vector fields V and W are said to be orthogonal if $V(p) \cdot W(p) = 0$ for all p. For example, V × W is orthogonal to V and W.

(d) Similarly the vector fields E_1, E_2, E_3 are orthonormal if

$$E_i \cdot E_j = \delta_{ij},$$

that is $(E_i \cdot E_i)(p) = 1$ for all p and for i = 1, 2, 3 and E_i, E_j are orthogonal for $i \neq j$.

(e) (Lemma 6.3). If E_1, E_2, E_3 are orthonormal vector fields, any vector field V has an orthonormal expansion of the form

$$\begin{aligned} V &= (V \cdot E_1)E_1 + (V \cdot E_2)E_2 + (V \cdot E_3)E_3 \\ &= f_1E_1 + f_2E_2 + f_3E_3 \end{aligned}$$

where $f_i = V \cdot E_i$.

Moreover, if $W = g_1E_1 + g_2E_2 + g_3E_3$, we find, as one would expect, that

$$V \cdot W = f_1g_1 + f_2g_2 + f_3g_3.$$

In future we shall not always trouble to point out when the pointwise principle is used to make definitions or deduce results about vector fields. So long as you are sufficiently familiar with the linear algebra of E^3 you should have no trouble with adapting this to vector fields.

Summary

Notation

$V \cdot W$	Page 82, line 1
$V \times W$	Text, page 55
$\lVert V \rVert$	Page 82, line 2
E_1, E_2, E_3	Page 82, Definition 6.1
r, ϑ, z	Page 82, Example 6.2.(1)
ρ, ϑ, φ	Page 83, Example 6.2.(2)

Definitions

(i)	Orthogonal vector fields	Text, page 55
(ii)	Orthonormal vector fields	Text, page 55
(iii)	Frame field E_1, E_2, E_3	Page 82, Definition 6.1
(iv)	Coordinate functions with respect to a frame field	Page 83, Lemma 6.3

Examples

(i)	The natural frame field U_1, U_2, U_3	Page 82, second paragraph
(ii)	The cylindrical frame field E_1, E_2, E_3	Page 82, Example 6.2.(1)
(iii)	The spherical frame field F_1, F_2, F_3	Page 83, Example 6.2.(2)

Results

(i) For the cylindrical frame field

$$E_1 = \cos\vartheta\ U_1 + \sin\vartheta\ U_2$$

$$E_2 = -\sin\vartheta\ U_1 + \cos\vartheta\ U_2$$

$$E_3 = U_3.$$

Page 82, Example 6.2.(1)

(ii) For the spherical frame field

$$F_1 = \cos\varphi\ \cos\vartheta\ U_1 + \cos\varphi\ \sin\vartheta\ U_2 + \sin\varphi\ U_3$$

$$F_2 = -\sin\vartheta\, U_1 + \cos\vartheta\, U_2$$

$$F_3 = -\sin\varphi \cos\vartheta\, U_1 - \sin\varphi \sin\vartheta\, U_2 + \cos\varphi\, U_3.$$

Page 83, Example 6.2.(2)

(iii) If E_1, E_2, E_3 is a frame field and V is a vector field, then

$$V = (V \cdot E_1)E_1 + (V \cdot E_2)E_2 + (V \cdot E_3)E_3.$$

Page 83, Lemma 6.3.(1)

(iv) If E_1, E_2, E_3 is a frame field, $V = \Sigma f_i E_i$ and $W = \Sigma g_i E_i$, then

$$V \cdot W = \Sigma f_i g_i.$$

Page 83, Lemma 6.3.(2)

Techniques

(i)	Recognition of frame fields.	Page 82, Definition 6.1
(ii)	Production of frame fields using Gram-Schmidt and cross product (see Lemma II.1.A).	
(iii)	Expression of a vector field in terms of a frame field, and applications thereof.	Page 82, Example 6.2

Exercises

Techniques (i) and (ii)

1. Page 84, Exercise 1.

Technique (iii)

2. Page 84, Exercise 2.

Solutions

1. Page 84, Exercise 1.

First we notice that $\|V\|$ and $\|\widetilde{W}\|$ are never zero and so E_1 and E_2 are well-defined. Since V and W are linearly independent at each p, we know that

$$V(p) \neq 0$$

$$W(p) \neq 0$$

$$W(p) \neq \frac{(W\cdot V)(p)}{\|V(p)\|^2} V(p).$$

$V(p) \neq 0$ and so $\|V\|$ is never zero.

$$\begin{aligned}\widetilde{W} &= W - (W\cdot E_1)E_1\\ &= W - \frac{(W\cdot V)V}{\|V\|^2},\end{aligned}$$

which is also never zero.

We must show that $E_i\cdot E_j = \delta_{ij}$.

$$E_1\cdot E_1 = \frac{V}{\|V\|}\cdot\frac{V}{\|V\|} = \frac{V\cdot V}{\|V\|^2} = \frac{\|V\|^2}{\|V\|^2} = 1;$$

$$E_2\cdot E_2 = \frac{\widetilde{W}}{\|\widetilde{W}\|}\cdot\frac{\widetilde{W}}{\|\widetilde{W}\|} = 1 \quad \text{similarly.}$$

$$\begin{aligned}E_1\cdot E_2 &= \frac{V}{\|V\|}\cdot\frac{\widetilde{W}}{\|\widetilde{W}\|} = \frac{V\cdot\widetilde{W}}{\|V\|\,\|\widetilde{W}\|} = \frac{V\cdot(W - W\cdot E_1\, E_1)}{\|V\|\,\|\widetilde{W}\|}\\ &= \frac{V\cdot W - (W\cdot E_1)(V\cdot E_1)}{\|V\|\,\|\widetilde{W}\|} = \frac{\|V\|(E_1\cdot W) - (W\cdot E_1)(\|V\|E_1\cdot E_1)}{\|V\|\,\|\widetilde{W}\|}\\ &= \frac{\|V\|\,(W\cdot E_1) - \|V\|(W\cdot E_1)}{\|V\|\,\|\widetilde{W}\|} = 0.\end{aligned}$$

Since $E_3 = E_1 \times E_2$, $E_1\cdot E_3 = E_2\cdot E_3 = 0$ and

$$\begin{aligned}E_3\cdot E_3 &= ((E_1\cdot E_1)(E_2\cdot E_2) - (E_1\cdot E_2)^2)^{\frac{1}{2}}, \qquad \text{by Lemma II.1.8,}\\ &= 1.\end{aligned}$$

2. Page 84, Exercise 2.

(a) $$\begin{aligned}U_1 &= \cos\vartheta\, E_1 - \sin\vartheta\, E_2\\ &= \cos\varphi\,\cos\vartheta\, F_1 - \sin\vartheta\, F_2 - \sin\varphi\,\cos\vartheta\, F_3.\end{aligned}$$

(b) $$\begin{aligned}\cos\vartheta\, U_1 + \sin\vartheta\, U_2 + U_3 &= E_1 + E_3\\ &= (\cos\varphi + \sin\varphi)F_1 + (\cos\varphi - \sin\varphi)F_3.\end{aligned}$$

(c) (i) $$xU_1 + yU_2 + zU_3 = r\cos\vartheta\, U_1 + r\sin\vartheta\, U_2 + zU_3$$
$$= r\,E_1 + z\,U_3;$$

(ii) $$xU_1 + yU_2 + zU_3 = \rho\cos\varphi\cos\vartheta\, U_1 + \rho\cos\varphi\sin\vartheta\, U_2 + \rho\sin\varphi\, U_3$$
$$= \rho F_1.$$

If you cannot do this type of exercise by inspection, you may find the following method helpful. We can write the equations giving the F_i in terms of the U_i in matrix form as

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\vartheta & \cos\varphi\sin\vartheta & \sin\varphi \\ -\sin\vartheta & \cos\vartheta & 0 \\ -\sin\varphi\cos\vartheta & -\sin\varphi\sin\vartheta & \cos\varphi \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = M \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix};$$

Since the F_i and the U_i are both frames, M must be an orthogonal matrix, i.e. $M^{-1} = {}^tM$, so we can write $\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = {}^tM \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$, giving the U_i in terms of the F_i.

As an example, we work part (b)(ii):

$$\cos\vartheta\, U_1 + \sin\vartheta\, U_2 + U_3 = (\cos\vartheta, \sin\vartheta, 1)\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}$$

$$= (\cos\vartheta, \sin\vartheta, 1)\,{}^tM \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

$$= (\cos\vartheta, \sin\vartheta, 1)\begin{pmatrix} \cos\varphi\cos\vartheta & -\sin\vartheta & -\sin\varphi\cos\vartheta \\ \cos\varphi\sin\vartheta & \cos\vartheta & -\sin\varphi\sin\vartheta \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

$$= (\cos\varphi + \sin\varphi, 0, \cos\varphi - \sin\varphi)\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

$$= (\cos\varphi + \sin\varphi)F_1 + (\cos\varphi - \sin\varphi)F_3.$$

II.7 CONNECTION FORMS

Introduction

These last two sections of Chapter II are of quite a different nature to the preceding ones. The material is more abstract, and concerned almost entirely with forms. We expect you will find these sections difficult. However, they are not needed again until Chapter VI, so you may be content with just a superficial understanding at this stage of the course. If necessary, you can revise these sections when you reach Chapter VI: you should find them easier going at that reading after the lapse of time.

Section II.7 draws together the previous two sections and Section I.5. Its purpose is to generalize the Frenet formulas to arbitrary frame fields. The matrix techniques introduced look messy but are simple to use in practice.

READ: Section II.7 (pages 85–90).

Comment

(i) *Page 88, Theorem 7.3* The ij-th entry of $dA\,{}^tA = \sum_k (dA)_{ik}({}^tA)_{kj} = \sum_k (da_{ik})(a_{jk})$, which we write as $\sum_k a_{jk}da_{ik}$ to keep to the standard convention of putting the differentials on the right. We cannot use such a conventional arrangement with the matrices dA and ${}^tA$, however, as matrix multiplication is not commutative.

Supplementary Comment

(i) *Page 89: dA* The fact that $d(\cos\vartheta) = -\sin\vartheta\, d\vartheta$, and the other similar results, follow from Lemma I.5.7.

Summary

Notation

ω_{ij}	Page 85, Lemma 7.1
ω	Page 87, line 4
A	Page 88, line 11
dA	Page 88, line 18

Definitions

(i)	Connection forms ω_{ij}	Page 85, Lemma 7.1
(ii)	Attitude matrix (A) of a frame field	Page 88, line 11
(iii)	Differential of a matrix whose entries are functions from E^3 to R	Page 88, line 18

Results

(i)	ω_{ij} is a 1-form and $\omega_{ij} = -\omega_{ji}$.	Page 85, Lemma 7.1
(ii)	The connection equations: $\nabla_V E_i = \Sigma\omega_{ij}(V)E_j$.	Page 86, Theorem 7.2
(iii)	$\omega = dA\,{}^tA$.	Page 88, Theorem 7.3

Technique

(i) Calculation of the connection forms using $\omega = dA\,{}^tA$.

Page 88, Theorem 7.3

Exercises

Technique (i)

1. Page 91, Exercise 4, using A and ω.

Theory Exercise

2. Page 91, Exercise 5.

Solutions

1. Page 91, Exercise 4.

$$A = \begin{pmatrix} \cos\varphi\cos\vartheta & \cos\varphi\sin\vartheta & \sin\varphi \\ -\sin\vartheta & \cos\vartheta & 0 \\ -\sin\varphi\cos\vartheta & -\sin\varphi\sin\vartheta & \cos\varphi \end{pmatrix}$$

$$dA = \begin{pmatrix} -\sin\varphi\cos\vartheta\, d\varphi - \cos\varphi\sin\vartheta\, d\vartheta & -\sin\varphi\sin\vartheta\, d\varphi + \cos\varphi\cos\vartheta\, d\vartheta & \cos\varphi\, d\varphi \\ -\cos\vartheta\, d\vartheta & -\sin\vartheta\, d\vartheta & 0 \\ -\cos\varphi\cos\vartheta\, d\varphi + \sin\varphi\sin\vartheta\, d\vartheta & -\cos\varphi\sin\vartheta\, d\varphi - \sin\varphi\cos\vartheta\, d\vartheta & -\sin\varphi\, d\varphi \end{pmatrix}$$

$$\omega = dA^tA = \begin{pmatrix} 0 & \cos\varphi\, d\vartheta & d\varphi \\ -\cos\varphi\, d\vartheta & 0 & \sin\varphi\, d\vartheta \\ -d\varphi & -\sin\varphi\, d\vartheta & 0 \end{pmatrix}$$

so $\omega_{12} = \cos\varphi\, d\vartheta$

$\omega_{13} = d\varphi$

$\omega_{23} = \sin\varphi\, d\vartheta$.

2. Page 91, Exercise 5.

$$\nabla_V W = \nabla_V\Big(\sum_i f_iE_i\Big)$$

$$= \sum_i \nabla_V(f_iE_i), \qquad \text{by Corollary II.5.4.(1),}$$

$$= \sum_i (V[f_i]E_i + f_i\nabla_V E_i), \qquad \text{by Corollary II.5.4.(3),}$$

$$= \sum_i \Big(V[f_i]E_i + f_i\sum_j \omega_{ij}(V)E_j\Big), \qquad \text{by the connection equations,}$$

$$= \sum_i V[f_j]E_j + \sum_j\sum_i f_i\omega_{ij}(V)E_j, \qquad \text{by rearrangement,}$$

$$= \sum_j \Big\{V[f_j] + \sum_i f_i\omega_{ij}(V)\Big\}E_j.$$

In the case of the natural frame field all the connection forms are zero (prove!) and so this equation reduces to

$$\nabla_V W = \sum_j V[f_j]U_j$$

as given on page 79.

II.8 THE STRUCTURAL EQUATIONS

Introduction

As mentioned in the previous section, you will probably find this section rather abstract and hard going. Do not attempt it until you feel fairly confident about both Section II.6 and Section II.7. You should also be familiar with the ideas of *linear functional, dual space* and *dual basis* (see *Unit M201 12, Linear Functionals and Duality,* Sections 1 and 2).

This section deals with finding the exterior derivatives of the connection forms introduced in Section II.7 and of the *dual 1-forms* of a frame field. As we do not yet know what an exterior derivative is, this section, like I.6, must be approached purely formally, as manipulation of strings of symbols. In Chapter IV we shall find a meaning for the exterior derivative, and in Chapter VI we shall relate that to the results developed here.

The results of this section are not needed until Chapter VI. You will probably find it hard: if so, do not worry about understanding it in depth now, but return to it from time to time so that it becomes fairly familiar before Chapter VI is reached.

READ: Section II.8 (pages 91–95).

Comments

(i) *Page 91: dual 1-forms* In Lemma II.6.3 we saw that if E_1, E_2, E_3 is a frame field then any vector field V can be expressed as

$$V = (V \cdot E_1)E_1 + (V \cdot E_2)E_2 + (V \cdot E_3)E_3.$$

In a similar manner, we obtain the fact that, if v_p is a tangent vector,

$$v_p = (v_p \cdot E_1(p))E_1(p) + (v_p \cdot E_2(p))E_2(p) + (v_p \cdot E_3(p))E_3(p),$$

simply by applying Lemma II.1.5 to the frame $E_1(p)$, $E_2(p)$, $E_3(p)$. Thus if we define the functions θ_i from the set of all tangent vectors to E^3 to R (and extend to vector fields by the pointwise principle) by

$$\theta_i(v_p) = v_p \cdot E_i(p)$$

we may write

$$v_p = \theta_1(v_p)E_1(p) + \theta_2(v_p)E_2(p) + \theta_3(v_p)E_3(p)$$

$$V = \theta_1(V)E_1 + \theta_2(V)E_2 + \theta_3(V)E_3.$$

θ_i is certainly linear at each point, and so θ_i is a 1-form. In fact, since the E_i are orthonormal, we have

$$\theta_i(E_j) = \delta_{ij}$$

which is precisely what is meant by saying that the θ_i are the dual basis to the E_i. This is why we call the θ_i the dual 1-forms of the frame field.

(ii) *Page 94: Example* The θ_i have not been found yet, but a method for so doing is given on page 95. The ω_{ij} were computed in Exercise II.7.4.

Summary

Notation

θ_i	Page 91, Definition 8.1

Definition

(i)	Dual 1-form θ_i.	Page 91, Definition 8.1

Results

(i)	If V is a vector field, then $V = \Sigma\theta_i(V)E_i$.	Text, page 61
(ii)	$\theta_i(E_j) = \delta_{ij}$,	Text, page 61
(iii)	If ϕ is a 1-form, then $\phi = \Sigma\phi(E_i)\theta_i$.	Page 92, Lemma 8.2
(iv)	If $E_i = \sum_j a_{ij}U_j$, then $\theta_i = \sum_j a_{ij}\, dx_j$.	Page 92, line 20
(v)	The structural equations $d\theta_i = \sum_j \omega_{ij} \wedge \theta_j$ $d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}$.	Page 92, Theorem 8.3

Techniques

(i)	Finding the dual 1-forms θ_i by adhoc methods or using the attitude matrix.	Page 92, line 20
(ii)	Using the structural equations.	Page 92, Theorem 8.3

Exercises

Technique (i)

1. Page 96, Exercise 4. Part (c) depends on knowing what $\partial f/\partial r$, $\partial f/\partial\vartheta$ mean and on knowing the formula

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial\vartheta}d\vartheta + \frac{\partial f}{\partial z}dz.$$

These will be explained in the final section of this text, so you may like to defer this part of the exercise until then.

Technique (ii)

2. Check the structural equations for the cylindrical frame field. (The connection forms were calculated on pages 89–90.)

Theory Exercise

3. Page 95, Exercise 1.

Solutions

1. Page 96, Exercise 4.

(a) $E_1 = \cos\vartheta\ U_1 + \sin\vartheta\ U_2$

$E_2 = -\sin\vartheta\ U_1 + \cos\vartheta\ U_2$

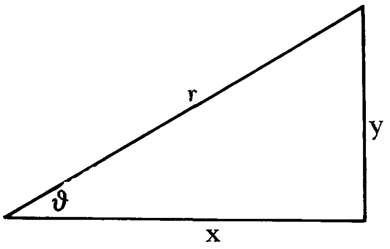
$E_3 = U_3.$

By definition, $\theta_1(U_j) = U_j\cdot E_1 = E_1\cdot U_j$, so

$$\theta_1(U_1) = \cos\vartheta \qquad \theta_1(U_2) = \sin\vartheta \qquad \theta_1(U_3) = 0.$$

We compare this with the value of dr on each of the U_i.

$$dr(U_i) = U_i[r] = \frac{\partial r}{\partial x_i} \qquad \text{by Lemma I.3.2.}$$



$$r = (x^2 + y^2)^{\frac{1}{2}}$$

so

$$\frac{\partial r}{\partial x} = \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} = \frac{x}{r} = \cos\vartheta$$

$$\frac{\partial r}{\partial y} = \frac{y}{(x^2 + y^2)^{\frac{1}{2}}} = \frac{y}{r} = \sin\vartheta$$

$$\frac{\partial r}{\partial z} = 0.$$

Thus

$$dr(U_1) = \cos\vartheta = \theta_1(U_1)$$

$$dr(U_2) = \sin\vartheta = \theta_2(U_2)$$

$$dr(U_3) = 0 \quad = \theta_3(U_3).$$

Since the 1-forms θ_1 and dr are equal on the frame field U_i they are equal on all tangent vectors and vector fields (by Lemma I.5.4), and so θ_1 = dr.

Similarly $\theta_2(U_j) = U_j\cdot E_2$, so

$$\theta_2(U_1) = -\sin\vartheta \qquad \theta_2(U_2) = \cos\vartheta \qquad \theta_2(U_3) = 0.$$

Also

$$d\vartheta(U_i) = U_i[\vartheta] = \frac{\partial\vartheta}{\partial x_i}.$$

$$\vartheta = \tan^{-1}\left(\frac{y}{x}\right)$$

so

$$\frac{\partial\vartheta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{-y}{x^2} = \frac{-y}{x^2+y^2} = \frac{-r\sin\vartheta}{r^2} = \frac{-\sin\vartheta}{r}$$

$$\frac{\partial\vartheta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\vartheta}{r^2} = \frac{\cos\vartheta}{r}$$

$$\frac{\partial\vartheta}{\partial z} = 0.$$

Thus

$$r\,d\vartheta(U_1) = -\frac{r\sin\vartheta}{r} = \theta_2(U_1)$$

$$r\,d\vartheta(U_2) = \frac{r\cos\vartheta}{r} = \theta_2(U_2)$$

$$r\,d\vartheta(U_3) = 0 = \theta_2(U_3)$$

and so $\theta_2 = r\,d\vartheta$.

$$\theta_3(U_1) = 0 \qquad \theta_3(U_2) = 0 \qquad \theta_3(U_3) = 1$$

$$dz(U_1) = 0 \qquad dz(U_2) = 0 \qquad dz(U_3) = 1$$

and so $\theta_3 = dz$.

(b) $E_i[r] = dr(E_i) = \theta_1(E_i) = \delta_{1i}$.

$$E_i[\vartheta] = d\vartheta(E_i) = \frac{\theta_2(E_i)}{r} = \frac{\delta_{2i}}{r}.$$

$$E_i[z] = dz(E_i) = \theta_3(E_i) = \delta_{3i}.$$

(c) $E_i[f] = df(E_i) = \left(\frac{\partial f}{\partial r}\,dr + \frac{\partial f}{\partial\vartheta}\,d\vartheta + \frac{\partial f}{\partial z}\,dz\right)(E_i)$ by the theorem to be proved in the final section,

$$= \frac{\partial f}{\partial r}\,\delta_{1i} + \frac{1}{r}\frac{df}{\partial\vartheta}\,\delta_{2i} + \frac{\partial f}{\partial z}\,\delta_{3i}.$$

Notice how we have used the Kronecker delta in (b) and (c) to convey several pieces of information in one statement. It does not matter if your solution did not use it, but bear in mind that you can often shorten the amount of writing by using it.

2. Now we check the structural equations.

$\theta_1 = dr$, $\theta_2 = r\,d\vartheta$, $\theta_3 = dz$,

$\omega_{12} = d\vartheta$, $\omega_{13} = 0$, $\omega_{23} = 0$ as found in Section II.7:

$d\theta_1 = d(dr) = 0$ and $\omega_{12}\wedge\theta_2 + \omega_{13}\wedge\theta_3 = d\vartheta\wedge r\,d\vartheta + 0 = 0$;

$d\theta_2 = d(r\,d\vartheta) = dr\wedge d\vartheta$ and $\omega_{21}\wedge\theta_1 + \omega_{23}\wedge\theta_3 = -d\vartheta\wedge dr + 0 = dr\wedge d\vartheta$;

$d\theta_3 = d(dz) = 0$ and $\omega_{31}\wedge\theta_1 + \omega_{32}\wedge\theta_2 = 0$;

$d\omega_{12} = d(d\vartheta) = 0$ and $\omega_{13}\wedge\omega_{32} = 0$;

$d\omega_{13} = 0$ and $\omega_{12}\wedge\omega_{23} = 0$;

$d\omega_{23} = 0$ and $\omega_{21}\wedge\omega_{13} = 0$.

3. Page 95, Exercise 1.

$$\begin{aligned}
d\phi &= d\Big(\sum_i f_i\theta_i\Big) \\
&= \sum_i d(f_i\theta_i) \\
&= \sum_i (df_i \wedge \theta_i + f_i d\theta_i), && \text{by Theorem I.6.4.(2),} \\
&= \sum_i \Big(df_i \wedge \theta_i + f_i \sum_j \omega_{ij} \wedge \theta_j\Big), && \text{by Theorem II.8.3.(1),} \\
&= \sum_j df_j \wedge \theta_j + \sum_j \sum_i f_i\omega_{ij} \wedge \theta_j, && \text{by rearrangement,} \\
&= \sum_j \Big\{df_j + \sum_i f_i\omega_{ij}\Big\} \wedge \theta_j.
\end{aligned}$$

PARTIAL DIFFERENTIATION

We have met two new coordinate systems in this chapter: r, ϑ, z and ρ, ϑ, φ. If you have met these before you may have come across expressions such as $\partial f/\partial r$, $\partial f/\partial\vartheta$ and equations such as

$$df = \frac{\partial f}{\partial\rho}\, d\rho + \frac{\partial f}{\partial\vartheta}\, d\vartheta + \frac{\partial f}{\partial\varphi}\, d\varphi.$$

Since the only partial derivatives we have so far formally defined are the $\partial f/\partial x_i$, where x_i are the natural coordinate functions, our purpose here is to give a meaning to expressions such as $\partial f/\partial r$, $\partial f/\partial\vartheta$ which is consistent with our idea of a partial derivative.

To do this properly, we must first of all be quite clear about the use of the symbols x, y, z, as defined on page 4. They denote not "dummy variables", but the natural coordinate function from E^3 to R. That is, if $p = (p_1, p_2, p_3)$ then $x(p) = p_1$, $y(p) = p_2$, $z(p) = p_3$. In particular $(x, y, z)(p) = (p_1, p_2, p_3) = p$, and so (x, y, z) is the identity function on E^3.

Now suppose that f is a function from E^3 to R. We must first notice that, although f is a *single* function from E^3 to R, it has different expressions in terms of x, y, z and in terms of ρ, ϑ, φ say. For example, let us consider the function $f = x^2 + y^2 + z^2$. We know $\rho^2 = x^2 + y^2 + z^2$, so we can also write $f = \rho^2$. Now, because (x, y, z) is the identity function, we can write

$$f = f \circ (x, y, z) = f(x, y, z)$$

but f is *not* equal to $f(\rho, \vartheta, \varphi)$ because

$$f(\rho, \vartheta, \varphi) = f \circ (\rho, \vartheta, \varphi) = \rho^2 + \vartheta^2 + \varphi^2.$$

Can we find a function g such that $f = g(\rho, \vartheta, \varphi)$, that is $f = g \circ m$ where m is the function $(\rho, \vartheta, \varphi)\colon E^3 \longrightarrow E^3$?

$$\begin{tikzcd}
E^3 \arrow[rr, "f"] \arrow[dr, "{m=(\rho,\vartheta,\varphi)}"'] & & R \\
& E^3 \arrow[ur, "g"'] &
\end{tikzcd}$$

In this case we can do so quite simply. If $p \in E^3$ then

$$f(p) = \rho^2(p) = (\rho(p))^2$$

$$g(\rho, \vartheta, \varphi)(p) = g(\rho(p), \vartheta(p), \varphi(p))$$

and so we require the function g to pick out the first coordinate and square it: but the function that does this is precisely the function we have called x^2. Thus $g = x^2$ and so

$$f = x^2 \circ m.$$

We return to the general case, and recall that the definition of $\partial f/\partial x$ is

$$\frac{\partial f}{\partial x}(p) = \frac{d}{dt} f(t, p_2, p_3)\Big|_{t=p_1} = \frac{d}{dt} f(t, y(p), z(p))\Big|_{t=x(p)}.$$

Roughly speaking, we are holding y and z fixed and differentiating with respect to x. If we want to define $\partial f/\partial \rho$ in a similar fashion we need to hold ϑ and φ fixed and differentiate with respect to ρ. This is where we need to use the function g: since

$$f = g(\rho, \vartheta, \varphi)$$

we define $\dfrac{\partial f}{\partial \rho}$ by

$$\frac{\partial f}{\partial \rho}(p) = \frac{d}{dt} g(t, \vartheta(p), \varphi(p))\Big|_{t=\rho(p)}.$$

But this is precisely $\partial g/\partial x$ evaluated at $(\rho(p), \vartheta(p), \varphi(p))$, because, roughly speaking, $\partial g/\partial x$ means "differentiate with respect to the first variable" no matter what the "variables" may be called.

This is formalized in the following definition.

Definition

If $m = (m_1, m_2, m_3)$ is a differentiable one-to-one function from a subset of E^3 to E^3 with differentiable inverse m^{-1}, and f is a differentiable function from E^3 to R, and g is a function such that $f = g \circ m$ (in fact g must be $f \circ m^{-1}$), then

$$\frac{\partial f}{\partial m_i}(p) = \frac{\partial g}{\partial x_i}(m(p)).$$

Notice that this definition of $\partial f/\partial m_i$ is dependent on the whole function m: if $\bar{m} = (m_1, \bar{m}_2, \bar{m}_3)$ then $\partial f/\partial m_1$ may be different depending on whether m or $\bar{m}$ is being used.

In practice this definition is very easy to work with. In our previous example we had $f = \rho^2$, $g = x^2$, and so $\partial f/\partial \rho = (\partial g/\partial x)(m) = 2x(\rho, \vartheta, \varphi) = 2\rho$. Another example: if $f = \rho \cos\varphi$ then $g = x \cos z$, and

$$\frac{\partial f}{\partial \rho} = \frac{\partial g}{\partial x}(m) = \cos z(\rho, \vartheta, \varphi) = \cos\varphi;$$

$$\frac{\partial f}{\partial \vartheta} = \frac{\partial g}{\partial y}(m) = 0;$$

$$\frac{\partial f}{\partial \varphi} = \frac{\partial g}{\partial z}(m) = -x \sin z\,(\rho, \vartheta, \varphi) = -\rho \sin\varphi.$$

This should show you that this process effectively treats ρ, ϑ, φ as "dummy variables" just as x, y, z are normally treated, and so the new partial derivatives can be written down without the step of introducing g.

For the usual partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$ we had the result (Corollary I.5.5):

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

Now that we have defined other partial derivatives suitably, we can prove an analogous theorem.

Theorem

If $m = (m_1, m_2, m_3)$ is a differentiable one-to-one function from a subset of E^3 to E^3 with differentiable inverse m^{-1}, and f is a differentiable function from E^3 to R, then

$$df = \Sigma \frac{\partial f}{\partial m_i} dm_i.$$

In particular,

$$df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \vartheta} d\vartheta + \frac{\partial f}{\partial z} dz;$$

$$df = \frac{\partial f}{\partial \rho} d\rho + \frac{\partial f}{\partial \vartheta} d\vartheta + \frac{\partial f}{\partial \varphi} d\varphi.$$

(Corollary I.5.5 is the particular case of this theorem for $m_i = x_i$.)

Proof

We use the chain rule given in the additional text on Section I.7. We can take $g = f \circ m^{-1}$ to obtain

$$f = g \circ m.$$

Now the chain rule gives

$$f_* = g_*(m)m_*.$$

If v is a tangent vector at the point p, this gives

$$f_*(v) = g_*(m(p))m_*(v), \qquad (*)$$

where g_* is written as a matrix evaluated at m(p) and the vectors $f_*(v)$, $m_*(v)$ are written as column vectors. Now, Theorem I.7.5 tells us that

$$f_*(v) = (v[f]) = (df(v)):$$

this is a 1 × 1 matrix so that corresponding column vector is also (df(v)). The same theorem tells us that

$$\begin{aligned} m_*(v) &= (v[m_1], v[m_2], v[m_3]) \\ &= (dm_1(v), dm_2(v), dm_3(v)), \end{aligned}$$

which is

$$\begin{pmatrix} dm_1(v) \\ dm_2(v) \\ dm_3(v) \end{pmatrix}$$

as a column vector.

By Corollary I.7.7 g_* has matrix

$$\left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3} \right).$$

Evaluating this at the point m(p) we have

$$\left(\frac{\partial g}{\partial x_1}(m(p)), \frac{\partial g}{\partial x_2}(m(p)), \frac{\partial g}{\partial x_3}(m(p))\right)$$

which is precisely

$$\left(\frac{\partial f}{\partial m_1}(p), \frac{\partial f}{\partial m_2}(p), \frac{\partial f}{\partial m_3}(p)\right)$$

by the definition of $(\partial f/\partial m_i)$. Putting the expressions we have obtained into (*):

$$\begin{aligned}(df(v)) &= \left(\frac{\partial f}{\partial m_1}(p), \frac{\partial f}{\partial m_2}(p), \frac{\partial f}{\partial m_3}(p)\right)\begin{pmatrix} dm_1(v) \\ dm_2(v) \\ dm_3(v)\end{pmatrix} \\ &= \left(\frac{\partial f}{\partial m_1}(p)dm_1(v) + \frac{\partial f}{\partial m_2}(p)dm_2(v) + \frac{\partial f}{\partial m_3}(p)dm_3(v)\right) \\ &= \left(\left(\frac{\partial f}{\partial m_1}dm_1 + \frac{\partial f}{\partial m_2}dm_2 + \frac{\partial f}{\partial m_3}dm_3\right)(v)\right).\end{aligned}$$

This is true for all tangent vectors **v**, and so

$$df = \frac{\partial f}{\partial m_1}dm_1 + \frac{\partial f}{\partial m_2}dm_2 + \frac{\partial f}{\partial m_3}dm_3.$$

You do not need to know any details of this proof. All you need for this course is the defintion of general partial derivatives and the result stated in the theorem.

Summary

Notation

$\left\{\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \vartheta}, \frac{\partial f}{\partial z}\right\}$	Text, page 66
$\left\{\frac{\partial f}{\partial \rho}, \frac{\partial f}{\partial \vartheta}, \frac{\partial f}{\partial \varphi}\right\}$	Text, page 66

Definition

(i)	Partial derivatives	Text, page 67

Results

(i)	$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \vartheta}d\vartheta + \frac{\partial f}{\partial z}dz.$	Text, page 68
(ii)	$df = \frac{\partial f}{\partial \rho}d\rho + \frac{\partial f}{\partial \vartheta}d\vartheta + \frac{\partial f}{\partial \varphi}d\varphi.$	Text, page 68

Techniques

(i)	Finding partial derivatives.	Text, page 67
(ii)	Using the expressions for df in terms of the partial derivatives.	Text, page 68

Exercises

Technique (i)

1. Find $\frac{\partial f}{\partial \rho}, \frac{\partial f}{\partial \vartheta}, \frac{\partial f}{\partial \varphi}$ when $f = \rho \cos\vartheta \cos\varphi$.

2. Find $\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \vartheta}, \frac{\partial f}{\partial z}$ when $f = x^2 + z^2$.

Technique (ii)

This has already been tested in Exercise II.8.4(c). Do that exercise now if you previously deferred it.

Solutions

1. $f = \rho\cos\vartheta \cos\varphi$

$= x \cos y \cos z \; (\rho, \vartheta, \varphi)$

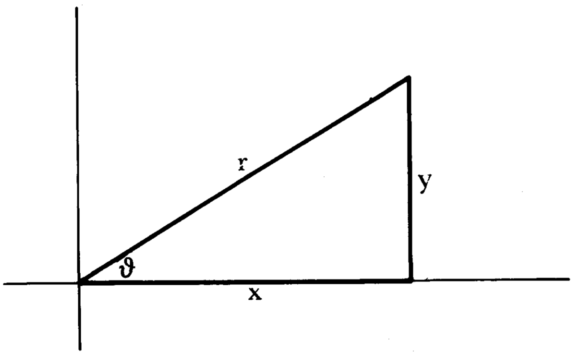
$= g(\rho, \vartheta, \varphi)$ where $g = x \cos y \cos z$.

$$\frac{\partial f}{\partial \rho} = \frac{\partial g}{\partial x}(\rho, \vartheta, \varphi) = \cos y \cos z \; (\rho, \vartheta, \varphi) = \cos\vartheta \cos\varphi.$$

$$\frac{\partial f}{\partial \vartheta} = \frac{\partial g}{\partial y}(\rho, \vartheta, \varphi) = -x \sin y \cos z \; (\rho, \vartheta, \varphi) = -\rho \sin\vartheta \cos\varphi.$$

$$\frac{\partial f}{\partial \varphi} = \frac{\partial g}{\partial z}(\rho, \vartheta, \varphi) = -x \cos y \sin z \; (\rho, \vartheta, \varphi) = -\rho \cos\vartheta \sin\varphi.$$

2. We must first of all express f in terms of r, ϑ, z.



From the diagram we see that

$x = r \cos\vartheta$

and so

$$f = x^2 + z^2$$
$$= r^2 \cos^2 \vartheta + z^2.$$

As before, we can introduce $g = x^2 \cos^2 y + z^2$, or we can go directly to the results:

$$\frac{\partial f}{\partial r} = 2r \cos^2 \vartheta,$$

$$\frac{\partial f}{\partial \vartheta} = -2r^2 \cos \vartheta \ \sin \vartheta;$$

$$\frac{\partial f}{\partial z} = 2z.$$

FURTHER EXERCISES AND SOLUTIONS

Section II.1

Recommended Exercises

Page 49, Exercise 2. This shows that Euclidean distance is a metric in the sense of M202 and M231.

Page 50, Exercise 11. This relates the vector operation grad to the norm and dot product.

Solutions

2(a). $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\| = (\Sigma(p_i - q_i)^2)^{\frac{1}{2}}$, which is non-negative.

$d(\mathbf{p}, \mathbf{q}) = 0 \Rightarrow \Sigma(p_i - q_i)^2 = 0$, and a sum of squares is zero if and only if each term is zero, that is $p_i = q_i$.

2(b). $(p_i - q_i)^2 = (q_i - p_i)^2$ so $d(\mathbf{p}, \mathbf{q}) = d(\mathbf{q}, \mathbf{p})$.

2(c). $d(\mathbf{p}, \mathbf{q}) + d(\mathbf{q}, \mathbf{r}) = \|\mathbf{p} - \mathbf{q}\| + \|\mathbf{q} - \mathbf{r}\| \geqslant \|\mathbf{p} - \mathbf{q} + \mathbf{q} - \mathbf{r}\| = \|\mathbf{p} - \mathbf{r}\| = d(\mathbf{p}, \mathbf{r})$.

11(a). $df(\mathbf{v}) = \Sigma\, v_i \dfrac{\partial f_i}{\partial x_i} = (\Sigma v_i U_i(\mathbf{p}))\cdot(\Sigma \dfrac{\partial f_i}{\partial x_i} U_i(\mathbf{p})) = \mathbf{v}\cdot\nabla f(\mathbf{p})$.

11(b). Put $\mathbf{w} = (\nabla f)(\mathbf{p})$.

$\mathbf{u}[f] \quad = \mathbf{u}\cdot\nabla f(\mathbf{p}) = \mathbf{u}\cdot\mathbf{w}$

and $|\mathbf{u}\cdot\mathbf{w}| \leqslant \|\mathbf{u}\|\,\|\mathbf{w}\| = \|\mathbf{w}\|$

with equality when $\mathbf{u} = \dfrac{\mathbf{w}}{\|\mathbf{w}\|}$.

Section II.2

Technique Exercise

Page 55, Exercise 1.

Other Recommended Exercises

Page 55, Exercise 5. This deal with unit-speed reparametrizations in the sense of Theorem II.2.1.

Page 56, Exercise 10. This shows that arc-length is preserved by reparametrizations which are symmetric in the sense of Comment (i).

Solutions

1(a). $(2, 2t, t^2)_{\alpha(t)}$, $(2, 2, 1)_{(2, 1, \frac{1}{3})}$;

$t^2 + 2$, 3;

$(0, 2, 2t)_{\alpha(t)}$, $(0, 2, 2)_{(2, 1, \frac{1}{3})}$.

1(b). $s(t) = 2t + \frac{t^3}{3}$; $\frac{14}{3}$.

5. If β_i is the unit-speed reparametrization based at $t = t_i$, and s_i is the arc-length function based at $t = t_i$,

$$\alpha(t) = \beta_1(s_1(t)) = \beta_2(s_2(t)).$$

$$\begin{aligned} s_1(t) &= \int_{t_1}^{t} v(u)du \\ &= \int_{t_1}^{t_2} v(u)du + \int_{t_2}^{t} v(u)du \\ &= s_0 + s_2(t) \end{aligned}$$

where s_0 is the arc-length of α from t_1 to t_2.

10. Put $\beta = \alpha(h)$.

(a) Arc-length of β from a to b

$$\begin{aligned} &= \int_a^b v_\beta(u)du \\ &= \int_{h^{-1}(c)}^{h^{-1}(d)} h'(u)v_\alpha(h(u))du \\ &= \int_c^d v_\alpha(t)dt \end{aligned}$$

= arc-length of α from c to d.

(b) Arc-length of β from a to b

$$\begin{aligned} &= \int_a^b v_\beta(u)du \\ &= \int_{h^{-1}(d)}^{h^{-1}(c)} (-h'(u))v_\alpha(h(u))du \\ &= -\int_d^c v_\alpha(t)dt \\ &= \int_c^d v_\alpha(t)dt. \end{aligned}$$

Section II.3

Technique Exercises

Page 63–64, Exercises 2, 3.

Other Recommended Exercises

Page 64, Exercise 5. This expresses all the Frenet formulas in a single format.

Page 64, Exercise 6. This deals with the approximation of a curve by part of a circle.

Solutions

2.
$$T(s) = \left(\frac{(1+s)^{\frac{1}{2}}}{2}, \frac{-(1-s)^{\frac{1}{2}}}{2}, \frac{1}{\sqrt{2}}\right)$$

$$N(s) = \left(\frac{(1-s)^{\frac{1}{2}}}{\sqrt{2}}, \frac{(1+s)^{\frac{1}{2}}}{\sqrt{2}}, 0\right)$$

$$B(s) = \left(\frac{-(1+s)^{\frac{1}{2}}}{2}, \frac{(1-s)^{\frac{1}{2}}}{2}, \frac{1}{\sqrt{2}}\right)$$

$$\kappa(s) = \tau(s) = \frac{1}{2\sqrt{2}(1-s^2)^{\frac{1}{2}}}$$

3. Evaluate both sides of each equation.

5. $A \times T = \kappa B \times T = \kappa N$

$A \times N = \tau T \times N + \kappa B \times N = \tau B - \kappa T$

$A \times B = \tau T \times B = -\tau N.$

6. The only solution for γ is

$$\gamma(s) = c - r\cos\frac{s}{r}N_0 + r\sin\frac{s}{r}T_0$$

where $c = \dfrac{N_0}{\kappa_0} + \beta(0)$

$$r = \frac{1}{\kappa_0}.$$

Section II.4

Technique Exercises

Pages 74–75, Exercises 1(a), 3(a), 4 (first part), 6, 7, 9(a), (b).

Other Recommended Exercises

Page 74, Exercise 5. This gives a formula for κ.

Page 75, Exercise 13. This introduces another special curve derived from a given curve. Since one of a, $\bar{a}$ is negative, you will first have to adapt Example II.3.3 for the case where a is negative.

Solutions

1(a). $T(t) = \frac{1}{2+t^2}(2, 2t, t^2)$

$N(t) = \frac{1}{2+t^2}(-2t, 2-t^2, 2t)$

$B(t) = \frac{1}{2+t^2}(t^2, -2t, 2)$

$\kappa(t) = \tau(t) = \frac{2}{(2+t^2)^2}.$

3(a). $T(0) = \frac{(1, 0, 1)}{\sqrt{2}}$

$N(0) = (0, 1, 0)$

$B(0) = \frac{(-1, 0, 1)}{\sqrt{2}}$

$\kappa(0) = 1$

$\tau(0) = \frac{3}{4}.$

4. Evaluate both sides of each equation.

6. $\alpha' = cT.$

$\alpha'' = cT' = c^2\kappa N.$

$\alpha' \times \alpha'' = c^3\kappa B.$

Now substitute in Theorem II.4.3.

7. $T(t) = \left(-\frac{a}{c}\sin t, \frac{a}{c}\cos t, \frac{b}{c}\right)$

$N(t) = (-\cos t, -\sin t, 0)$

$B(t) = \left(\frac{b}{c}\sin t, -\frac{b}{c}\cos t, \frac{a}{c}\right)$

$\kappa(t) = \frac{a}{c^2}$

$\tau(t) = \frac{b}{c^2}.$

9(a). $\mathbf{u} = \frac{(1, 0, 1)}{\sqrt{2}}$

$\vartheta = \frac{\pi}{4}$

$\gamma(t) = \left(t - \frac{t^3}{6}, t^2, -t + \frac{t^3}{6}\right).$

9(b). $\mathbf{u} = (0, 0, 1)$

$\vartheta = \frac{\pi}{4}$

$\gamma(t) = (3t - t^3, 3t^2, 0).$

5. $$\alpha'' = \frac{dv}{dt}T + \kappa v^2 N$$

$$\|\alpha''\|^2 = \left(\frac{dv}{dt}\right)^2 + (\kappa v^2)^2.$$

13. If $a > 0$, then $\kappa(s) = \frac{a}{c^2}$, $N(s) = (-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0)$;

if $a < 0$, then $\kappa(s) = -\frac{a}{c^2}$, $N(s) = (\cos\frac{s}{c}, \sin\frac{s}{c}, 0)$.

Substitute these to obtain

$$\beta_{ab}{}^* = \beta_{\bar{a}b}$$

for all a. Now $(\bar{\bar{a}}) = a$ so

$$\beta_{\bar{a}b}{}^* = \beta_{ab}.$$

Section II.5

Technique Exercises

Page 80, Exercises 1(b), 2(b), (d), (e), (f), 4.

Other Recommended Exercises

Page 80, Exercise 3, This extends the result that if Y is a vector field on a curve and $\|Y\|$ is constant then $Y \cdot Y' = 0$.

Page 80, Exercise 6(b). This deals with the definition of covariant derivative.

Page 80, Exercise 7. This exercise introduces the bracket of two vector fields.

Solutions

1(b). $(1, 2, 4)_p$

2(b). $-y\, U_3$.

2(d). $-\sin x\; U_1 + \cos x\; U_3$.

2(e). $-y^2 \cos x\; U_1 - y^2 \sin x\; U_2$.

2(f). $(y^2 - x\cos x - yz\sin x)U_1 + (-x\sin x + yz\cos x)U_2 - 2xy\, U_3$.

4. $X = \Sigma x_i U_i$.

$$\nabla_V X = \sum_i V[x_i]U_i = \sum_i \sum_j v_j \frac{\partial x_i}{\partial x_j} U_i = \sum_i \sum_j v_j \delta_{ij} U_i$$

$$= \sum_i v_i U_i = V.$$

3. $W(p+tv) \cdot W(p+tv) = \text{constant}$

$\Rightarrow W(p+tv) \cdot W(p+tv)' = 0$

$\Rightarrow W(p) \cdot \nabla_v W = 0.$

6(b). $W(\alpha)'(t) = \nabla_{\alpha'(t)} W = \nabla_v W$.

7(a). Both sides equal $\sum_i \sum_j \left(v_j \dfrac{\partial w_i}{\partial x_j} - w_j \dfrac{\partial v_i}{\partial x_j} \right) \dfrac{\partial f}{\partial x_i}$.

7(b). The definition is antisymmetric.

7(c). $$[U,[V,W]][f] = U[V,W][f] - [V,W]U[f]$$
$$= UVW[f] - UWV[f] - VWU[f] + WVU[f].$$

Adding the other two similar expressions to this gives zero.

7(d). $$[fV,gW][h] = fV[gW[h]] - gW[fV[h]]$$
$$= f(V[g]W[h] + gVW[h]) - g(W[f]V[h] + fWV[h])$$
$$= fV[g]W[h] - gW[f]V[h] + fg[V,W][h].$$

Section II.6

Techinque Exercises

Pages 84–85, Exercises 3, 4.

Solutions

3. Choose any unit E_2 orthogonal to E_1 and put $E_3 = E_1 \times E_2$. For example

$$E_2 = \sin z\ U_2 - \cos z\ U_3$$
$$E_3 = -\sin x\ U_1 + \cos x \cos z\ U_2 + \cos x \sin z\ U_3.$$

4. The same formulas are obtained as for the spherical frame field.

Section II.7

Technique Exercises

Pages 90–91, Exercises 1, 2, 3, 7.

Solutions

1. $\omega_{12} = 0;\ \omega_{13} = \omega_{23} = \dfrac{df}{\sqrt{2}}$.

2. All zero.

3. A is orthogonal.

$\omega_{12} = -df;$

$\omega_{13} = \cos f\ df;$

$\omega_{23} = \sin f\ df.$

7(a). For any f,

$$F_1[f] = \cos\varphi \cos\vartheta \frac{\partial f}{\partial x} + \cos\varphi \sin\vartheta \frac{\partial f}{\partial y} + \sin\varphi \frac{\partial f}{\partial z}.$$

$$\rho = (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

$$\vartheta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\varphi = \tan^{-1}\left(\frac{z}{(x^2+y^2)^{\frac{1}{2}}}\right):$$

evaluate the partial derivatives and substitute.

7(b). $-\sin\rho\, F_2 + \cos\rho\, F_3$.

Section II.8

No further exercises are recommended.

M334 II

DIFFERENTIAL GEOMETRY

I **Calculus on Euclidean Space**

II **Frame Fields**

III **Euclidean Geometry**

IV **Calculus on a Surface**

V **Shape Operators**

VI **Geometry of Surfaces in E^3**

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34 PART III

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Mathematics: A Third Level Course

PART III EUCLIDEAN GEOMETRY

DIFFERENTIAL GEOMETRY

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Mathematics: A Third Level Course

DIFFERENTIAL GEOMETRY PART III

EUCLIDEAN GEOMETRY

A commentary on Chapter III of O'Neill's
Elementary Differential Geometry
Prepared by the Course Team

THE OPEN UNIVERSITY PRESS

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1.2

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III.5 Congruence of Curves 32

Further Exercises and Solutions 38

Set Book

Barrett O'Neill: *Elementary Differential Geometry* (Academic Press, 1966). It is essential to have this book: the course is based on it and will not make sense without it.

Bibliography

The set books for M201, M231 and MST 282 are referred to occasionally; they are useful but not essential. They are:

D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins: *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

E.D. Nering: *Linear Algebra and Matrix Theory* (John Wiley, 1970).

M. Spivak: *Calculus,* paperback edition, (W.A. Banjamin/Addison-Wesley 1973).

R.C. Smith and P. Smith: *Mechanics,* SI edition (John Wiley, 1972).

Conventions

Before starting work on this text, please read M334 Part Zero. Consult the Errata List and the Stop Press and make any necessary alterations for this chapter in the set book.

Unreferenced pages and sections denote the set book. Otherwise

O'Neill	denotes the set book;
Text	denotes the correspondence text;
KKOP	denotes *An Introduction to Linear Analysis* by D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins;
Nering	denotes *Linear Algebra and Matrix Theory by* E.D. Nering;
Spivak	denotes *Calculus* by M. Spivak;
Smith	denotes *Mechanics* by R.C. Smith and P. Smith.

References to Open University Courses in Mathematics take the form:

Unit M100 22, Linear Algebra 1

Unit MST 281 10, Taylor Approximation

Unit M201 16, Euclidean Spaces I: Inner Products

Unit M231 2, Functions and Graphs

Unit MST 282 1, Some Basic Tools.

III.1 INTRODUCTION AND ISOMETRIES OF E^3

Introduction

The aim of this chapter is to prove that a unit-speed curve in E^3 is determined, apart from its position in space, by its curvature and torsion functions, which were introduced in Section II.3; in other words, that the curvature and torsion of a curve, as functions of the arc-length, tell us all there really is to know about it. We cannot begin to prove this until we have decided what we mean by "apart from position in space": when are two curves in different positions in E^3 going to be declared "essentially the same"? We shall not finally answer this question until Section III.5, but we make a start in this section by defining *isometries*, that is mappings of E^3 which preserve distance. You have met some isometries before — translations in *Unit M201 15, Affine Geometry and Convex Cones*, Section 1.1, and orthogonal linear transformations in *Unit M201 24, Orthogonal and Symmetric Transformations*, Section 1. You should remind yourself of the results of the latter section now, as they are used in this section. Apart from that, this section depends only on Sections I.7 and II.1; the later sections of Chapter II are not referred to yet.

READ: Introduction to Chapter III and Section III.1 (pages 98–102).

Comments

(i) *Isometries* An important point about the isometries of E^3 is that they form a group; that is, that they satisfy the following conditions.

(a) The composition of two isometries is again an isometry — this is Lemma 1.3.

(b) Composition of isometries is associative — this is true for composition of any functions.

(c) There is an identity isometry I such that IF = FI = F for all isometries F — this is true because the identity mapping I is an isometry.

(d) Each isometry F has an inverse F^{-1} which is also an isometry. This condition requires slightly more checking. It is an immediate consequence of Definition 1.1 that isometries are one-to-one mappings, for if F(p) = F(q) then

$$d(p, q) = d(F(p), F(q)) = d(F(p), F(p)) = 0$$

and so p = q. The fact that isometries of E^3 are also onto mappings follows from Theorem 1.7, because the translation T is onto and the orthogonal transformation C is onto, so their composite F = TC must also be onto. Since F is one-to-one and onto F has an inverse function $F^{-1} = C^{-1}T^{-1}$, which is a mapping because both C^{-1} and T^{-1} are mappings (see Comments (ii) and (iii)). Now

$$d(F^{-1}(p), F^{-1}(q)) = d(FF^{-1}(p), FF^{-1}(q)) \quad \text{because F is an isometry}$$

$$= d(p, q) \quad \text{because } FF^{-1} = I$$

and so F^{-1} is indeed an isometry.

(ii) *Translations* The translations form an important set of isometries, as noted in Example 1.2(1). We shall use the notation T_a to denote translation by a, that is,

$$T_a(p) = p + a.$$

Then

$$T_bT_a(p) = p + a + b = T_{a+b}(p) = T_{b+a}(p)$$

so T_bT_a is translation by b + a. In fact the translations correspond to the points of E^3 in such a way that the composition of translations corresponds to ordinary vector addition; hence the commutativity of composition of translations follows from the commutativity of vector addition. We have the following commutative diagram:

$$\begin{tikzcd}
a, b \arrow[r, "+"] \arrow[d, "\text{corresponding translation}"'] & a + b \arrow[d, "\text{corresponding translation}"] \\
T_a, T_b \arrow[r, "\text{composition}"] & T_aT_b = T_{a+b}
\end{tikzcd}$$

Since the identity mapping I is translation through 0, $T_a^{-1} = T_{-a}$ and we have this commutative diagram also:

$$\begin{tikzcd}
a \arrow[r, "\text{additive inverse}"] \arrow[d, "\text{corresponding translation}"'] & -a \arrow[d, "\text{corresponding translation}"] \\
T_a \arrow[r, "\text{inverse}"] & (T_a)^{-1} = T_{-a}
\end{tikzcd}$$

Thus the translations also form a group.

(iii) *Orthogonal Transformations* As noted in Lemma 1.5, these form another important set of isometries. The definition given on page 100 is in fact equivalent to that given in *Unit M201 24, Orthogonal and Symmetric Transformations*, page 8, because all the spaces in which we are interested are finite-dimensional. Thus we can make use of the result proved in that unit, that the following conditions on a linear transformation C of finite-dimensional Euclidean spaces are all equivalent:

(a) C preserves dot products (that is, C is an orthogonal transformation);

(b) The rows of the matrix C are orthonormal;

(c) The columns of the matrix C are orthonormal;

(d) ${}^tCC = C{}^tC = I$; that is, $C^{-1} = {}^tC$ (that is, C is an orthogonal matrix);

(e) C maps some orthonormal basis to another orthonormal basis;

(f) C maps all orthonormal bases to orthonormal bases.

Since, for orthogonal transformations C_1, C_2, C,

$$ {}^t(C_1C_2) = {}^tC_2{}^tC_1 = C_2^{-1}C_1^{-1} = (C_1C_2)^{-1} $$

and

$$ {}^t(C^{-1}) = {}^t({}^tC) = C = (C^{-1})^{-1} $$

this immediately tells us that the composite of two orthogonal transformations is an orthogonal transformation and that the inverse of an orthogonal transformation is an orthogonal transformation. Thus these also form a group.

(iv) *Page 101: Theorem 1.7* This breakdown of an isometry F as TC is very useful. In future work with isometries we shall nearly always use this expression, sometimes without explicitly defining T and C. If $F_1 = T_1C_1$ and $F_2 = T_2C_2$ are isometries then their composite F_1F_2 is also an isometry (Lemma 1.3) so we can write $F_1F_2 = T_3C_3$, where T_3 is the translation part of F_1F_2 and C_3 is the orthogonal part of F_1F_2. Then

$$ T_3C_3 = T_1C_1T_2C_2 $$

but it is *not* generally true that $T_3 = T_1T_2$ (although $C_3 = C_1C_2$) because C_1 and T_2 do not commute in general. (See the Theory Exercises if you are interested in investigating this further.)

Supplementary Comments

(i) *Page 99: rotation* The formulas for the coordinates of q in terms of those of p were worked out in *Unit M201 24, Orthogonal and Symmetric Transformations*, Section 1.2.

(ii) *Page 101: lines −10 and −9* If we put $\mathbf{r} = a\mathbf{p} + b\mathbf{q}$, then $r_i = ap_i + bq_i$ for i = 1, 2, 3. Using the identity established in line −11,

$$ F(\mathbf{r}) = \Sigma r_iF(\mathbf{u}_i), \qquad F(\mathbf{p}) = \Sigma p_iF(\mathbf{u}_i), \qquad F(\mathbf{q}) = \Sigma q_iF(\mathbf{u}_i) $$

so

$$ \begin{aligned} F(\mathbf{r}) &= \Sigma r_iF(\mathbf{u}_i) \\ &= \Sigma (ap_i + bq_i)F(\mathbf{u}_i) \\ &= \Sigma ap_iF(\mathbf{u}_i) + \Sigma bq_iF(\mathbf{u}_i) \\ &= a\Sigma p_iF(\mathbf{u}_i) + b\Sigma q_iF(\mathbf{u}_i) \\ &= aF(\mathbf{p}) + bF(\mathbf{q}). \end{aligned} $$

Summary

Notation

F	Page 98, Definition 1.1
T	Page 98, Example 1.2(1)
T_a	Text, page 6
C	Page 100, line 11
I	Page 99, line −6

Definitions

(i)	Isometry F	Page 98, Definition 1.1
(ii)	Translation by a, T_a	Page 98, Example 1.2(1)
(iii)	Orthogonal transformation C	Page 100, line 10
(iv)	Identity mapping I	Page 99, line −6

Example

(i)	Rotation about the z axis	Page 99, Example 1.2(2)

Results

(i)	The composite of two isometries is again an isometry.	Page 99, Lemma 1.3
(ii)	For translations:	
	$T_aT_b = T_bT_a = T_{a+b}$;	Page 99, Lemma 1.4(1)
	$(T_a)^{-1} = T_{-a}$.	Page 100, Lemma 1.4(2)
(iii)	Given p, q in E^3, there is a unique translation T such that $T(p) = q$.	Page 101, Lemma 1.4(3)
(iv)	If T is a translation and $T(p) = p$ for any p, then $T = I$.	Page 100, lines 8–9
(v)	Translations are isometries.	Page 98, Example 1.2(1)
(vi)	Orthogonal transformations are isometries.	Page 100, Lemma 1.5
(vii)	Any isometry fixing 0 is an orthogonal transformation.	Page 100, Lemma 1.6
(viii)	Each isometry F has a unique expression as TC, where T is a translation and C is an orthogonal transformation. Moreover $T = T_{F(0)}$.	Page 101, Theorem 1.7
(ix)	Every isometry has an inverse isometry.	Text, page 5.

Techniques

(i)	Recognition of isometries, translations, orthogonal transformations.	Page 98, Definition 1.1, Page 98, Example 1.2(1), and Page 100, line 10.
(ii)	Expression of an isometry in standard form TC.	Page 101, Theorem 1.7
(iii)	Calculation of the image of a point under an isometry expressed in standard form.	Page 102, line −5

Exercises

Technique (i)

1. Page 103, Exercise 4.
2. Page 103, Exercise 6, (a) – (d), first part.

Technique (ii)

3. Page 103, Exercise 6, second part.

Technique (iii)

4. Page 103, Exercise 5(a).

Theory Exercises (omit if short of time)

5. Page 103, Exercise 1.
6. Page 103, Exercise 2.
7. Page 103, Exercise 3.

Solutions

1. Page 103, Exercise 4.

As noted in Comment (iii), C is orthogonal $\Longleftrightarrow C^tC = I \Longleftrightarrow$ the rows of C are orthonormal. We check the last condition as it involves only six calculations.

$$\left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)\cdot\left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right) = \frac{1}{9}(4+4+1) = 1$$

$$\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)\cdot\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \frac{1}{9}(4+1+4) = 1$$

$$\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{1}{9}(1+4+4) = 1$$

$$\left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)\cdot\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \frac{1}{9}(-4+2+2) = 0$$

$$\left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{1}{9}(-2+4-2) = 0$$

$$\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{1}{9}(2+2-4) = 0$$

This shows that C is orthogonal.

$$C(p) = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -6 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{19}{3} \\ -\frac{7}{3} \end{pmatrix}$$

$$C(q) = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{3} \\ -\frac{4}{3} \\ \frac{7}{3} \end{pmatrix}$$

$p \cdot q = 3.1 + 1.0 + (-6)(3) = 3 - 18 = -15.$

$C(p) \cdot C(q) = \left(\frac{2}{3}\right)\left(-\frac{5}{3}\right) + \left(\frac{19}{3}\right)\left(-\frac{4}{3}\right) + \left(-\frac{7}{3}\right)\left(\frac{7}{3}\right) = \frac{1}{9}(-10 - 76 - 49) = -\frac{135}{9} = -15.$

2. Page 103, Exercise 6.

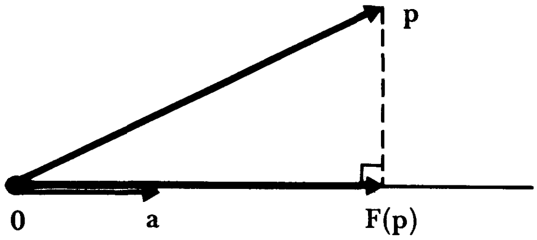
In each case we must check whether $d(F(p), F(q)) = d(p, q)$ for all p and q.

(a) $F(p) = -p.$

$$\begin{aligned} d(F(p), F(q)) &= d(-p, -q) \\ &= \|-p - (-q)\| \\ &= \|(-1)(p - q)\| \\ &= |-1| \, \|p - q\| \qquad \text{(using } \|ap\| = |a| \, \|p\|\text{)} \\ &= 1 \, \|p - q\| \\ &= d(p, q). \end{aligned}$$

This is true for all p and q, so F is an isometry.

(b) In this case $F\colon p \longmapsto p \cdot a \, a$ is just the perpendicular projection of E^3 onto the one-dimensional subspace spanned by a. We can easily see from the picture



that F does not preserve norms: for example, we can choose p to be orthogonal to a and then $F(p) = 0$ so $\|F(p)\| \neq \|p\|$. Since F does not preserve norms, F cannot preserve dot products.

Alternatively we proceed as follows.

$$\begin{aligned} d(F(p), F(q)) &= d(p \cdot a \, a, q \cdot a \, a) \\ &= \|p \cdot a \, a - q \cdot a \, a\| \\ &= \|(p \cdot a - q \cdot a)a\| \\ &= |p \cdot a - q \cdot a| \, \|a\|, && \text{as above,} \\ &= |p \cdot a - q \cdot a|, && \text{since } \|a\| = 1, \\ &= |(p - q) \cdot a| \\ &= \|p - q\| \, \|a\| \, |\cos \vartheta|, && \text{where } \vartheta \text{ is the angle between } p - q \text{ and } a, \\ &= d(p, q) \, |\cos \vartheta|, && \text{since } \|a\| = 1, \end{aligned}$$

which is not equal to $d(p, q)$ unless ϑ is a multiple of π.

As yet another approach, since the image of F is not the whole of E^3, F is not onto E^3 and so F cannot be an isometry, by Comment (i).

(c) $F(p) = (p_3 - 1, p_2 - 2, p_1 - 3)$.

$$\begin{aligned} d(F(p), F(q)) &= d((p_3 - 1, p_2 - 2, p_1 - 3), (q_3 - 1, q_2 - 2, q_1 - 3)) \\ &= \|(p_3 - 1, p_2 - 2, p_1 - 3) - (q_3 - 1, q_2 - 2, q_1 - 3)\| \\ &= \|(p_3 - q_3, p_2 - q_2, p_1 - q_1)\| \\ &= ((p_3 - q_3)^2 + (p_2 - q_2)^2 + (p_1 - q_1)^2)^{\frac{1}{2}} \\ &= \|(p_1 - q_1, p_2 - q_2, p_3 - q_3)\| \\ &= \|p - q\| \\ &= d(p, q). \end{aligned}$$

This is true for all p and q, and so F is an isometry.

(d) $F(p) = (p_1, p_2, 1)$.

$$\begin{aligned} d(F(p), F(q)) &= d((p_1, p_2, 1), (q_1, q_2, 1)) \\ &= \|(p_1, p_2, 1) - (q_1, q_2, 1)\| \\ &= \|(p_1 - q_1, p_2 - q_2, 0)\| \\ &= ((p_1 - q_1)^2 + (p_2 - q_2)^2)^{\frac{1}{2}}, \end{aligned}$$

which is not equal to d(p, q) unless $p_3 = q_3$. For example,

$$d((0, 0, 0), (0, 0, 1)) = 1 \quad \text{but}$$

$$d(F(0, 0, 0), F(0, 0, 1)) = d((0, 0, 1), (0, 0, 1)) = 0.$$

Thus F is not an isometry.

Again, we could note that, since the image of F is

$$\{ p \in E^3 : p_3 = 1 \} \neq E^3,$$

F is not onto, and so cannot be an isometry.

3. Page 103, Exercise 6.

(a) $F(p) = -p$ and so $F = -I$, which is an orthogonal transformation. This is already in standard form, and so T is the identity, $C = -I$.

(b) F is not an isometry.

(c) $F(0) = (-1, -2, -3)$ so T, the translation part of F, is translation by $(-1, -2, -3)$. If $F = TC$,

$$F(p) = TC(p)$$

i.e. $(p_3 - 1, p_2 - 2, p_1 - 3) = C(p) + (-1, -2, -3)$

i.e. $(p_3, p_2, p_1) + (-1, -2, -3) = C(p) + (-1, -2, -3)$

so $C(p) = (p_3, p_2, p_1)$

so C has matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(In fact this represents reflection in the plane $x = z$.)

(d) F is not an isometry.

4. Page 103, Exercise 5(a).

$F(p) = T_a C(p) = a + C(p)$

$$= \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} -\frac{6}{\sqrt{2}} \\ -2 \\ \frac{10}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 3\sqrt{2} \\ 1 \\ -1 + 5\sqrt{2} \end{pmatrix}$$

5. Page 103, Exercise 1.

Put $F = CT_a$. Since C and T_a are isometries F is an isometry (by Lemma III.1.3), and thus by Theorem III.1.7 we can write $F = T_b D$ where T_b is translation through b and D is an orthogonal transformation. Now

$$CT_a(p) = T_b D(p) \qquad \text{for all } p$$

i.e.

$$C(a + p) = b + D(p)$$

i.e.

$$C(a) + C(p) = b + D(p).$$

Putting $p = 0$ gives $C(a) = b$ because $C(0) = D(0) = 0$.

Substituting back, we have

$$b + C(p) = b + D(p) \qquad \text{for all } p$$

i.e.

$$C(p) = D(p) \qquad \text{for all } p$$

and so

$$C = D.$$

Thus

$$F = T_b D = T_{C(a)} C.$$

Alternatively, we could show that CT_a and $T_{C(a)}C$ have the same effect on every point of E^3. For p in E^3,

$$CT_a(p) = C(a + p) = C(a) + C(p),$$

while

$$T_{C(a)}C(p) = T_{C(a)}(C(p)) = C(a) + C(p).$$

6. Page 103, Exercise 2.

$$F = T_aA \quad \text{and} \quad G = T_bB, \quad \text{so}$$

$$FG = T_aAT_bB.$$

By the result of Exercise III.1.1,

$$AT_b = T_{A(b)}A$$

and so

$$FG = T_aAT_bB = T_aT_{A(b)}AB.$$

$T_aT_{A(b)}$ is the translation through a + A(b); AB is an orthogonal transformation, being the product of two orthogonal transformations. Thus FG has translation part $T_{a + A(b)}$ and orthogonal part AB.

Similarly GF has translation part $T_{b + B(a)}$ and orthogonal part BA.

7. Page 103, Exercise 3.

$F = T_aC$. Since T_a and C both have inverses we know that

$$F^{-1} = (T_aC)^{-1} = C^{-1}T_a^{-1}.$$

Now

$$\begin{aligned} C^{-1}T_a^{-1} &= C^{-1}T_{-a}, && \text{by Lemma III.1.4(2),} \\ &= T_{C^{-1}(-a)}C^{-1}, && \text{by Exercise III.1.1,} \\ &= T_{-C^{-1}(a)}C^{-1}, && \text{by linearity of } C^{-1}. \end{aligned}$$

Thus the translation part of F^{-1} is $T_{-C^{-1}(a)}$; the orthogonal part of F^{-1} is C^{-1}.

III.2 THE DERIVATIVE MAP OF AN ISOMETRY

Introduction

This section follows on from Sections III.1 and I.7. We also need to use certain results from M201 – that there exists a unique linear transformation taking one basis of E^3 to another basis of E^3 (see *Unit M201 2, Linear Transformations*, Section 1.8) and that a linear transformation is orthogonal if and only if it takes an orthonormal basis to another orthonormal basis (see *Unit M201 24, Orthogonal and Symmetric Transformations*, Section 1.2).

The purpose of this short section is to show that there is a unique isometry taking any given *frame* in E^3 to any other given frame in E^3. In order to do this we have to look at the derivative map F_*: we find that this takes a particularly simple form when F is an isometry, for if $F = TC$ then F_* is represented simply by C.

READ: Section III.2 (pages 104–106).

Comment

(i) *Page 105: lines -18 to -15* F_{*p} and C cannot be the same function, for they act on different spaces – $T_p(E^3)$ and E^3 respectively. However, $T_p(E^3)$ is isomorphic to E^3 via the "canonical" isomorphism $\phi_p\colon E^3 \longrightarrow T_p(E^3)$ given by

$$\phi_p\colon v \longmapsto v_p \qquad (v \in E^3).$$

To say that F_{*p} and C "differ only by the canonical isomorphisms of E^3", means, in effect, that the following diagram commutes; that is, that F_{*p} and C send vectors that correspond to each other via ϕ_p to vectors that correspond to each other via $\phi_{F(p)}$.

$$\begin{tikzcd}
E^3 \arrow[r, "C"] \arrow[d, "\phi_p"'] & E^3 \arrow[d, "\phi_{F(p)}"] \\
T_p(E^3) \arrow[r, "F_{*p}"] & T_{F(p)}(E^3)
\end{tikzcd}$$

$$\begin{tikzcd}
v \arrow[r, "C"] \arrow[d, "\phi_p"'] & Cv \arrow[d, "\phi_{F(p)}"] \\
v_p \arrow[r, "F_{*p}"] & F_{*p}(v_p) = Cv_{F(p)}
\end{tikzcd}$$

Supplementary Comment

(i) *Page 105, line 4* If $v = (v_1, v_2, v_3)$ then the ith coordinate of Cv is $\sum_j c_{ij}v_j$, so that

$$(Cv)_q = \sum_i \sum_j c_{ij} v_j U_i(q)$$

for any point q. In particular, putting $q = F(p)$ gives

$$(Cv)_{F(p)} = \sum_i \sum_j c_{ij} v_j U_i(F(p)).$$

But $F_*(v_p) = (Cv)_{F(p)}$, and so

$$F_*(v_p) = \sum_i\sum_j c_{ij}v_jU_i(F(p)) = \sum_{i,j} c_{ij}v_jU_i(F(p)).$$

However, $v_p = \sum_j v_jU_j(p)$, so this becomes

$$F_*\left(\sum_j v_jU_j(p)\right) = \sum_{i,j} c_{ij}v_jU_i(F(p)),$$

which can be abbreviated to the result given in line 4 by omitting the "p"s and writing $\bar{U}_i$ for $U_i(F)$.

Summary

Results

(i)	The derivative map of an isometry $F = TC$ sends the tangent vector **v** at the point **p** to the tangent vector Cv at the point $F(p)$.	Page 104, Theorem 2.1
(ii)	If F is an isometry, F_{*p} is an orthogonal transformation from $T_p(E^3)$ to $T_{F(p)}(E^3)$.	Page 105, Corollary 2.2
(iii)	Given two frames in E^3, there is a unique isometry taking one onto the other.	Page 105, Theorem 2.3

Techniques

(i)	Finding the derivative map of an isometry using Result (i).	Page 104, Theorem 2.1
(ii)	Finding the isometry mentioned in Result (iii), using the method employed in the proof of Theorem 2.3.	Page 105, line −3 of text

Exercises

Technique (i)

1. Page 106, Exercise 1.

Technique (ii)

2. Page 107, Exercise 5.

Theory Exercises (omit if short of time)

3. Page 106, Exercise 2. Write $F = T_aA$ and $G = T_bB$ and use the results of Exercise III.1.2, whose solution occurs on Text page 13, and Exercise III.1.3, whose solution occurs on Text page 13.
4. Page 107, Exercise 4(a). (HINT: **r** is in the plane through **p** orthogonal to **q** if and only if $(\mathbf{r} - \mathbf{p})\cdot\mathbf{q} = 0$.)

Solutions

1. Page 106, Exercise 1.

We can write $T = TI$ for any isometry T, where I is the identity mapping. In the case when T is itself a translation this expresses T in standard form, as I is an orthogonal transformation. Using Theorem III.2.1 we have

$$\begin{aligned} T_*(v) &= (Iv)_{T(p)} \\ &= v_{T(p)}, \qquad \text{because } Iv = v. \end{aligned}$$

Thus $T_*(v)$ has the same Euclidean coordinates as v; only the point of application has changed. In other words, $T_*(v)$ is parallel to v.

2. Page 107, Exercise 5.

The attitude matrix of the **e** frame is

$$A = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

and the attitude matrix of the **f** frame is

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

By the argument at the end of the text on page 106 we must have that

$$C = {}^tBA = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix}$$

and T is the translation through

$$q - C(p) = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{4}{3\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -\frac{4}{3} \\ 1 - \frac{2\sqrt{2}}{3} \end{pmatrix}$$

Thus the isometry F is given by

$$F(r) = \begin{pmatrix} 3 \\ -\frac{4}{3} \\ 1 - \frac{2\sqrt{2}}{3} \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

for all points $r = (r_1, r_2, r_3)$ of E^3.

Let us check that F really does carry the e frame into the f frame; that is, $F_*(e_{ip}) = f_{iq}$ for i = 1, 2, 3; that is, $F(p) = q$ and $Ce_i = f_i$ for i = 1, 2, 3.

$$F(p) = \begin{pmatrix} 3 \\ -\frac{4}{3} \\ 1 - \frac{2\sqrt{2}}{3} \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -\frac{4}{3} \\ 1 - \frac{2\sqrt{2}}{3} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{4}{3\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = q.$$

$$Ce_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = f_1$$

$$Ce_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = f_2$$

$$Ce_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = f_3$$

3. Page 106, Exercise 2.

If $F = T_a A$ and $G = T_b B$, where A, B are the orthogonal parts of F and G, then, by the result of Exercise III.1.2, GF has orthogonal part BA. Hence, by Theorem III.2.1, if v_p is any tangent vector,

$$(GF)_*(v_p) = (BAv)_{GF(p)}.$$

The same theorem gives

$$F_*(v_p) = Av_{F(p)}$$

and, applied to the isometry G and the tangent vector $F_*(v_p)$,

$$G_*(F_*(v_p)) = B(Av)_{G(F(p))},$$

that is

$$(G_*F_*)(v_p) = (BAv)_{GF(p)}.$$

Hence $(GF)_*(v_p) = (G_*F_*)(v_p)$ for all tangent vectors v_p, and so it follows that $(GF)_* = G_*F_*$.

By the result of Exercise III.1.3, F^{-1} has orthogonal part A^{-1}, and so Theorem III.2.1 gives

$$(F^{-1})_*(v_p) = (A^{-1}v)_{F^{-1}(p)}$$

for any tangent vector v_p. We must show that this is the same as $(F_*)^{-1}(v_p)$. We know that

$$F_*(v_p) = (Av)_{F(p)},$$

that is, F_* acts on the vector part of v_p by the orthogonal transformation A and on the point of application of v_p by F. Thus $(F_*)^{-1}$, which is the inverse of F_*, must act on the vector part of v_p by A^{-1} and on the point of application of v_p by F^{-1}; that is,

$$(F_*)^{-1}(v_p) = (A^{-1}v)_{F^{-1}(p)}.$$

Consequently

$$(F_*)^{-1}(v_p) = (F^{-1})_*(v_p)$$

for all v_p, and so $(F_*)^{-1} = (F^{-1})_*$.

4. Page 107, Exercise 4(a).

r is in the plane through p orthogonal to q if and only if

$$(r - p)\cdot q = 0.$$

C preserves dot products so this happens if and only if

$$(C(r - p))\cdot C(q) = 0.$$

Suppose T is translation by the vector a.

C is linear, so

$$\begin{aligned} C(r - p) &= C(r) - C(p) \\ &= C(r) + a - (C(p) + a), \\ &= F(r) - F(p). \end{aligned}$$

Thus

$$(C(r - p))\cdot C(q) = (F(r) - F(p))\cdot C(q),$$

which is zero if and only if F(r) is in the plane through F(p) orthogonal to C(q). Hence F carries the plane through p orthogonal to q to the plane through F(p) orthogonal to C(q).

III.3 ORIENTATION

Introduction

This section continues from Section III.2, also tying in ideas from *Unit M201 24, Orthogonal and Symmetric Transformations*, Sections 2.2 to 2.4. You may recall from that unit that frames in E^3 are of two distinct types, known as left-handed and right-handed, distinguished by which hand "best fits" the frames. Moreover, orthogonal transformations of E^3 also fall into two distinct sets: there are the *proper rotations* (that is, those transformations physically realizable in 3-space as a rotation about an axis) and the *improper rotations* (that is, those transformations which are not realizable as an actual movement in 3-space because they consist of a proper rotation followed by a reflection). It turns out that proper rotations preserve the "handedness" of frames, whereas improper rotations interchange left-handed and right-handed frames, and that the corresponding matrices have determinant +1 for a proper rotation and −1 for an improper rotation.

In this section we give a more mathematical definition of "right-handed" and "left-handed" and classify all isometries of E^3 (not just orthogonal transformations) in terms of their effect upon frames. We are able to state and prove the results quite precisely, in a form that makes them suitable for use in proving the important theorems of Section III.5.

READ: Sections III.3 (pages 107–110).

Comment

(i) *Page 107: Remark 3.1*

(1) In this case the attitude matrix is I, and det I = +1.

(2) Since $\mathbf{e}_3$ is a unit vector orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_2$, it must be either $\mathbf{e}_1 \times \mathbf{e}_2$ or $-(\mathbf{e}_1 \times \mathbf{e}_2)$. By the result of Exercise II.1.4(d), $\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2)$, which is +1 if $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$, −1 if $\mathbf{e}_3 = -(\mathbf{e}_1 \times \mathbf{e}_2)$.

(3) This is a restatement of part of Lemma II.1.A, on Text, Part II, page 7.

Supplementary Comments

(i) *Page 110: the determinant* Remember that a formal expansion of

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

yields

$$(v_2w_3 - v_3w_2)e_1 + (v_3w_1 - v_1w_3)e_2 + (v_1w_2 - v_2w_1)e_3.$$

(ii) *Page 110: proof of Lemma 3.5* If the frame is negatively oriented, the e_1 component of $v \times w$ is

$$\begin{aligned} v_2e_2 \times w_3e_3 + v_3e_3 \times w_2e_2 &= (v_2w_3 - v_3w_2)e_2 \times e_3 \\ &= (v_2w_3 - v_3w_2)(-e_1), && \text{by Remark 3.1(3),} \\ &= \epsilon\,(v_2w_3 - v_3w_2)e_1 \end{aligned}$$

because $\epsilon = -1$ when the frame is negatively oriented.

(iii) *Page 110: end of proof of Theorem 3.6*

$$\begin{aligned} F_*(v) \times F_*(w) &= \epsilon \begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}, && \text{by Lemma III.3.5,} \\ &= \epsilon\,((v_2w_3 - v_3w_2)e_1 + \ldots) \\ &= \epsilon\,((v_2w_3 - v_3w_2)F_*(U_1(p)) + \ldots) \\ &= \epsilon\,(F_*((v_2w_3 - v_3w_2)U_1(p) + \ldots)), && \text{because } F_* \text{ is linear,} \\ &= \epsilon \left(F_* \begin{vmatrix} U_1(p) & U_2(p) & U_3(p) \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \right) \\ &= \epsilon\, F_*(v \times w), && \text{by the definition of cross product,} \end{aligned}$$

and

$$\begin{aligned} \epsilon = e_1 \cdot e_2 \times e_3 &= F_*(U_1(p)) \cdot F_*(U_2(p)) \times F_*(U_3(p)) \\ &= (\operatorname{sgn} F)U_1(p) \cdot U_2(p) \times U_3(p), && \text{by Lemma III.3.2,} \\ &= \operatorname{sgn} F, && \text{because the frame } U_1(p),\ U_2(p),\ U_3(p) \text{ is positively oriented.} \end{aligned}$$

Summary

Notation

sgn F	Page 108, line 8

Definitions

(i)	Positively oriented Right-handed	Page 107, line −17
(ii)	Negatively oriented Left-handed	Page 107, line −16

(iii)	Sign of an isometry, sgn F	Page 108, line 6
(iv)	Orientation-preserving	Page 109, Definition 3.3
(v)	Orientation-reversing	Page 109, Definition 3.3

Results

(i)	The natural frame field is positively oriented.	Page 107, Remark 3.1(1)
(ii)	The frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is positively oriented if and only if $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$.	Page 107, Remark 3.1(2)
(iii)	Frenet frames are positively oriented.	Page 107, Remark 3.1(2)
(iv)	If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is a positively oriented frame, $\mathbf{e}_i = \mathbf{e}_j \times \mathbf{e}_k = -\mathbf{e}_k \times \mathbf{e}_j$ for (i, j, k) = (1, 2, 3) or (2, 3, 1) or (3, 1, 2).	Page 107, Remark 3.1(3)
(v)	$F_*(\mathbf{e}_1)\cdot F_*(\mathbf{e}_2) \times F_*(\mathbf{e}_3) = \operatorname{sgn} F\ \mathbf{e}_1\cdot\mathbf{e}_2 \times \mathbf{e}_3$.	Page 108, Lemma 3.2
(vi)	The coordinates of tangent vectors in terms of a positively oriented frame give the expected formula for the cross product; if the frame is negatively oriented the usual formula must be multiplied by -1.	Page 110, Lemma 3.5
(vii)	Orientation-preserving isometries preserve cross products; orientation-reversing isometries change their sign: $F_*(\mathbf{v} \times \mathbf{w}) = \operatorname{sgn} F\ F_*(\mathbf{v}) \times F_*(\mathbf{w})$.	Page 110, Theorem 3.6

Techniques

(i)	Distinguishing positively and negatively oriented frames.	Page 107, lines -17, -16
(ii)	Finding the sign of an isometry	Page 108, lines 6 to 8
(iii)	Finding the effect of an isometry on a cross product, using Result (vii).	Page 110, Theorem 3.6

Exercises

Technique (i)

1. Find the orientation of each of the frames in Exercise III.2.5 on page 107.

Technique (ii)

2. Find the sign of the isometry in Exercise III.2.5 and check that Lemma III.3.2 is satisfied.

Technique (iii)

3. Page 111, Exercise 3.

Theory Exercises (omit if short of time)

4. Page 111, Exercise 1. (HINT: Use the result of Exercise III.1.2.)
5. Page 111, Exercise 2.

Solutions

1. The attitude matrix of the **e** frame is

$$\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix},$$

whose determinant is

$$\frac{2}{3}\left(\frac{1}{3}\cdot\frac{2}{3} - \frac{2}{3}\cdot\left(-\frac{2}{3}\right)\right) + \frac{2}{3}\left(\frac{2}{3}\cdot\frac{1}{3} - \left(-\frac{2}{3}\right)\cdot\frac{2}{3}\right) + \frac{1}{3}\left(-\frac{2}{3}\cdot\left(-\frac{2}{3}\right) - \frac{1}{3}\cdot\frac{1}{3}\right)$$

$$= \frac{1}{27}(2(2+4) + 2(2+4) + (4-1))$$

$$= 1,$$

so the **e** frame is positively oriented. Alternatively (and more simply)

$$\mathbf{e}_1 \times \mathbf{e}_2 = \frac{1}{9}(2.2 - 1.1,\ 1.-2 - 2.2,\ 2.1 - 2.(-2))$$

$$= \frac{1}{9}(3, -6, 6) = \mathbf{e}_3$$

and so the **e** frame is positively oriented by Remark 3.1(2). Similarly,

$$\mathbf{f}_1 \times \mathbf{f}_2 = \frac{1}{\sqrt{2}}(0.0 - 1.1,\ 1.0 - 1.0,\ 1.1 - 0.0)$$

$$= \frac{1}{\sqrt{2}}(-1, 0, 1) = -\mathbf{f}_3$$

and so the **f** frame is negatively oriented.

2. Lemma III.3.2 is satisfied if $F_*(\mathbf{e}_1)\cdot F_*(\mathbf{e}_2) \times F_*(\mathbf{e}_3) = \operatorname{sgn} F\ \mathbf{e}_1\cdot\mathbf{e}_2 \times \mathbf{e}_3$. We found in the previous section that

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix}.$$

$$\operatorname{sgn} F = \det C = \frac{1}{\sqrt{2}}\left(\frac{1}{3}\cdot - \frac{1}{3\sqrt{2}} - \frac{2}{3}\cdot\frac{4}{3\sqrt{2}}\right) + 0 + \frac{1}{\sqrt{2}}\left(-\frac{2}{3}\cdot\frac{4}{3\sqrt{2}} - \frac{1}{3}\cdot\frac{1}{3\sqrt{2}}\right)$$

$$= \frac{1}{18}(-1 - 8 - 8 - 1) = -1.$$

Since sgn F = −1, Lemma III.3.2 predicts that the **e** frame and the F(**e**) frame (that is the **f** frame) have opposite orientations: we have already shown that this is the case in the previous exercise.

3. Page 111, Exercise 3.

We want to show that $C_*(\mathbf{v} \times \mathbf{w}) = \operatorname{sgn} C\, C_*(\mathbf{v}) \times C_*(\mathbf{w})$.

$$\mathbf{v} = (3, 1, -1)$$

$$\mathbf{w} = (-3, -3, 1)$$

so

$$\mathbf{v} \times \mathbf{w} = (1 - 3, 3 - 3, -9 + 3) = (-2, 0, -6).$$

C is its own orthogonal part, so C_* is represented by the same matrix as C.

$$C_*(\mathbf{v}) = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

$$C_*(\mathbf{w}) = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{11}{3} \\ -\frac{7}{3} \end{pmatrix}$$

$$C_*(\mathbf{v} \times \mathbf{w}) = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} \frac{10}{3} \\ \frac{8}{3} \\ -\frac{14}{3} \end{pmatrix}$$

$$C_*(\mathbf{v}) \times C_*(\mathbf{w}) = (-1, 3, 1) \times \left(-\frac{1}{3}, -\frac{11}{3}, -\frac{7}{3}\right)$$

$$= \left(-\frac{10}{3}, -\frac{8}{3}, \frac{14}{3}\right) = -C_*(\mathbf{v} \times \mathbf{w})$$

so the formula is checked if sgn C = −1.

However,

$$\operatorname{sgn} C = \det C = -\frac{2}{3}\left(\frac{1}{3}.\frac{2}{3} + \frac{2}{3}.\frac{2}{3}\right) + \frac{2}{3}\left(-\frac{2}{3}.\frac{1}{3} - \frac{2}{3}.\frac{2}{3}\right) - \frac{1}{3}\left(\frac{2}{3}.\frac{2}{3} - \frac{1}{3}.\frac{1}{3}\right)$$

$$= \frac{1}{27}(-2(2+4) + 2(-2-4) - (4-1))$$

$$= -1$$

and so the formula is checked.

4. Page 111, Exercise 1.

Let $F = T_aA$, $G = T_bB$. Then, by the result of Exercise III.1.2, FG has orthogonal part AB. Hence

$$\begin{aligned} \operatorname{sgn}(FG) &= \det(AB), \quad \text{by definition,} \\ &= (\det A)(\det B) \\ &= (\operatorname{sgn} F)(\operatorname{sgn} G), \text{ by definition of sgn F, sgn G.} \end{aligned}$$

Similarly

$$\begin{aligned} \operatorname{sgn}(GF) &= (\operatorname{sgn} G)(\operatorname{sgn} F) \\ &= (\operatorname{sgn} F)(\operatorname{sgn} G). \end{aligned}$$

Putting $G = F^{-1}$, gives

$$\operatorname{sgn}(FF^{-1}) = (\operatorname{sgn} F)(\operatorname{sgn} F^{-1}).$$

But $FF^{-1} = I$, the identity mapping, and so

$$(\operatorname{sgn} F)(\operatorname{sgn} F^{-1}) = \operatorname{sgn} I = +1:$$

therefore

$$\begin{aligned} \operatorname{sgn} F^{-1} &= \frac{1}{\operatorname{sgn} F} \\ &= \operatorname{sgn} F, \qquad \text{because } \operatorname{sgn} F = \pm 1. \end{aligned}$$

5. Page 111, Exercise 2.

Let G be an orientation-reversing isometry of E^3. We want to express G as H_0F, so we define F to be $H_0{}^{-1}G$. Then certainly

$$H_0F = H_0\, H_0{}^{-1}G = G$$

and so all that remains to be shown is that F is orientation-preserving. Now, by the result of the previous exercise,

$$\operatorname{sgn} G = \operatorname{sgn}(H_0F) = (\operatorname{sgn} H_0)(\operatorname{sgn} F).$$

Here G and H_0 are both orientation-reversing, and so

$$\operatorname{sgn} G = \operatorname{sgn} H_0 = -1;$$

hence sgn $F = +1$ and so F is orientation-preserving.

III.4 EUCLIDEAN GEOMETRY

Introduction

This section depends on Sections III.1 – III.3 and Section II.3. Now we are able to use the ideas and results of Sections III.1 – III.3 to begin to define what we mean by "the same, but for position in space, as . . .". Loosely speaking, we are going to declare two "objects" in E^3 (for example, curves or vector fields) to be "the same but for position in space" if there is an isometry carrying one into the other. In order that this definition agree with our intuitive ideas of "sameness", two conditions must hold: (a) the relation "is the same, but for position in space as" must be an equivalence relation (any sensible definition of "same" must lead to an equivalence relation); (b) any properties of the objects concerned that we regard as essential, such as the curvature of a curve, the derivative of a vector field, must be preserved by isometries, so that objects which are "the same" have the same essential properties. Condition (a) is easily checked: the relation is transitive because the composite of isometries is itself an isometry (Lemma III.1.3); it is reflexive because I is an isometry; it is symmetric because each isometry has an inverse isometry (Comment (i) on Section III.1). Condition (b) is the concern of this section, where it is shown that isometries preserve derivatives of vector fields on curves and the Frenet apparatus of unit-speed curves.

READ: Section III.4 (pages 112–115).

Additional Text

(i) *Arbitrary-Speed Curves and Isometries* While it is true that any mapping F preserves the velocity of a curve α in the sense that

$$F_*(\alpha') = (F(\alpha))'$$

(Theorem I.7.8), the speed of a curve is not generally preserved, because $\|\alpha'\|$ and $\|(F(\alpha))'\|$ are not necessarily the same. However, in the case where F is an isometry an argument similar to that given at the beginning of the proof of Theorem 4.2 shows that F does preserve speed: F preserves norms, as noted on page 105, and so

$$\begin{aligned} v_{F(\alpha)} &= \|(F(\alpha))'\| \\ &= \|F_*(\alpha')\| \\ &= \|\alpha'\| \\ &= v_\alpha . \end{aligned}$$

We have shown that α and $F(\alpha)$ have the same speed if F is an isometry. The following exercise shows that they have the same curvature, and plus-or-minus the same torsion.

1. Let α be an arbitrary-speed regular curve in E^3, F an isometry of E^3, $\beta = F(\alpha)$, and $\bar{\alpha}$ a unit-speed reparametrization of α, where $\alpha = \bar{\alpha}(s)$. Prove that $\kappa_\beta = \kappa_\alpha$ and $\tau_\beta = \operatorname{sgn} F\ \tau_\alpha$. (HINT: Use the above result to prove that $\bar{\beta} = F(\bar{\alpha})$ is a unit-speed reparametrization of β in the sense of Theorem II.2.1, and then use Theorem III.4.2.)

Supplementary Comments

(i) *Page 113: second half* As usual we are writing $F = TC$ and $C = (c_{ij})$.

(ii) *Page 114: fourth line of proof* T and $\bar{T}$ are defined by $T = \beta'$, $\bar{T} = \bar{\beta}'$, so the result follows using the fact that $\bar{\beta}' = (F(\beta))' = F_*(\beta')$ by Theorem I.7.8.

(iii) *Page 114: seventh line of proof* κ and $\bar{\kappa}$ are defined by $\kappa = \|\beta''\|$, $\bar{\kappa} = \|\bar{\beta}''\|$; $\bar{\beta}'' = F_*(\beta'')$ by Corollary III.4.1; $\|F_*(\beta'')\| = \|\beta''\|$ because F_* preserves norms.

(iv) *Page 114: eleventh line of proof* $(F_*(\beta''))/\kappa = F_*(\beta''/\kappa)$ because F_* is a linear transformation at each point.

Summary

Results

(i) Isometries preserve derivatives of vector fields on curves: $(F_*(Y))' = F_*(Y')$.

In particular, isometries preserve acceleration. Page 113, Corollary 4.1

(ii) Isometries preserve the speed of curves. Text, Page 26

(iii) Isometries preserve the Frenet apparatus of unit-speed curves, except that orientation-reversing isometries change the sign of the torsion and reverse the direction of the binormal:

$\bar{\kappa} = \kappa$ $\bar{T} = F_*(T)$

$\bar{\tau} = \operatorname{sgn} F\, \tau$ $\bar{N} = F_*(N)$

$\bar{B} = \operatorname{sgn} F\, F_*(B)$ Page 114, Theorem 4.2

(iv) A similar result for arbitrary-speed curves: if $\beta = F(\alpha)$, then $\kappa_\beta = \kappa_\alpha$ and $\tau_\beta = \operatorname{sgn} F\, \tau_\alpha$. Text, Page 26

Techniques

(i) Applying Result (i) to find the derivatives of vector fields of the form $F_*(Y)$ on a curve $F(\alpha)$. Page 113, Corollary 4.1

(ii) Using Result (iii) to find the Frenet apparatus of curves of the form $F(\alpha)$. Page 114, Theorem 4.2

Exercises

Technique (i)

2. Page 115, Exercise 2. (Remember that, because C is orthogonal, C_* has the same matrix as C.)

Technique (ii)

3. Page 115, Exercise 1.

Solutions

1. We have seen that α and β have the same speed, and consequently α and β have the same arc-length function s. Now

$$\bar\beta(s) = (F(\bar\alpha))(s) = F(\bar\alpha(s)) = F(\alpha) = \beta$$

and so $\bar\beta$ is a unit-speed reparametrization of β in the sense of Theorem II.2.1.

Hence

$$\kappa_\beta = \kappa_{\bar\beta}(s) \qquad \tau_\beta = \tau_{\bar\beta}(s)$$

by the definition of the curvature and torsion of an arbitrary-speed curve. Similarly

$$\kappa_\alpha = \kappa_{\bar\alpha}(s) \qquad \tau_\alpha = \tau_{\bar\alpha}(s).$$

However, $\bar\alpha$ is unit-speed and $\bar\beta = F(\bar\alpha)$ and so Theorem III.4.2. gives

$$\kappa_{\bar\beta} = \kappa_{\bar\alpha} \qquad \tau_{\bar\beta} = \operatorname{sgn} F\ \tau_{\bar\alpha}.$$

Hence

$$\kappa_\beta = \kappa_{\bar\beta}(s) = \kappa_{\bar\alpha}(s) = \kappa_\alpha$$

and

$$\tau_\beta = \tau_{\bar\beta}(s) = \operatorname{sgn} F\ \tau_{\bar\alpha}(s) = \operatorname{sgn} F\ \tau_\alpha.$$

2. Page 115, Exercise 2.

$$\bar\alpha(t) = C(\alpha(t)) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \\ 2t \end{pmatrix}$$

$$= \begin{pmatrix} -\cos t \\ \frac{1}{\sqrt{2}}(\sin t - 2t) \\ \frac{1}{\sqrt{2}}(\sin t + 2t) \end{pmatrix}$$

$\overline{Y}(t) = C_*(Y(t))$: since C is itself an orthogonal transformation C_* has the same matrix as C and so

$$C_*(Y(t)) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} t \\ 1 - t^2 \\ 1 + t^2 \end{pmatrix}$$

$$= \begin{pmatrix} -t \\ -\sqrt{2}t^2 \\ \sqrt{2} \end{pmatrix}$$

That is,

$$\overline{Y}(t) = (-t, -\sqrt{2}t^2, \sqrt{2}).$$

Thus

$$\overline{Y}'(t) = (-1, -2\sqrt{2}t, 0).$$

Now

$$Y'(t) = (1, -2t, 2t)$$

and so

$$C_*(Y'(t)) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ -2t \\ 2t \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -2\sqrt{2}t \\ 0 \end{pmatrix}$$

Thus $\overline{Y}'(t) = C_*(Y'(t))$ for all t, and so $\overline{Y}' = C_*(Y')$.

$$\alpha'(t) = (-\sin t, \cos t, 2),$$

$$\alpha''(t) = (-\cos t, -\sin t, 0)$$

so

$$C_*(\alpha''(t)) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos t \\ -\frac{1}{\sqrt{2}}\sin t \\ -\frac{1}{\sqrt{2}}\sin t \end{pmatrix}$$

whereas

$$\bar{\alpha}'(t) = \left(\sin t, \frac{1}{\sqrt{2}}(\cos t - 2), \frac{1}{\sqrt{2}}(\cos t + 2)\right)$$

$$\bar{\alpha}''(t) = \left(\cos t, -\frac{\sin t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}\right)$$

and so $C_*(\alpha''(t)) = \bar{\alpha}''(t)$ for all t; that is, $C_*(\alpha'') = \bar{\alpha}''$.

We have already computed Y', α'', $\bar{Y}'$, $\bar{\alpha}''$: from this we see that

$$\begin{aligned}(Y'\cdot\alpha'')(t) &= (1, -2t, 2t)\cdot(-\cos t, -\sin t, 0) \\ &= -\cos t + 2t\sin t\end{aligned}$$

and

$$\begin{aligned}(\bar{Y}'\cdot\bar{\alpha}'')(t) &= (-1, -2\sqrt{2}t, 0)\cdot\left(\cos t, \frac{-\sin t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}\right) \\ &= -\cos t + 2t\sin t\end{aligned}$$

so

$$Y'\cdot\alpha'' = \bar{Y}'\cdot\bar{\alpha}''.$$

3. Page 115, Exercise 1.

(a) Since β has unit speed, $T = \beta'$, and so if β is a cylindrical helix then there is a constant unit vector **u** and a constant angle ϑ such $\beta'\cdot\mathbf{u} = \cos\vartheta$. Let $\bar{\mathbf{u}} = F_*(\mathbf{u})$. Then

$$\|\bar{\mathbf{u}}\| = \|F_*(\mathbf{u})\| = \|\mathbf{u}\| = 1$$

and so $\bar{\mathbf{u}}$ is a constant unit vector. Moreover, $F(\beta)$ also has unit speed; that is, $\bar{T} = F(\beta)'$, and so

$$\begin{aligned}\bar{T}\cdot\bar{\mathbf{u}} &= (F(\beta))'\cdot\bar{\mathbf{u}} \\ &= (F(\beta))'\cdot F_*(\mathbf{u}) \\ &= F_*(\beta')\cdot F_*(\mathbf{u}), && \text{by Theorem I.7.8,} \\ &= \beta'\cdot\mathbf{u}, && \text{because } F_* \text{ preserves dot products,} \\ &= \cos\vartheta,\end{aligned}$$

and so $F(\beta)$ is also a cylindrical helix.

(b) Let $\overset{\frown}{F(\beta)}$ denote the spherical image of $F(\beta)$, $\bar{T}$ the unit tangent vector field on $F(\beta)$. Then $\overset{\frown}{F(\beta)}$ is the curve whose points have coordinates given by $\bar{T}$, while $\tilde{\beta}$ is the curve whose points have coordinates given by T. In other words

$$T(t) = \tilde{\beta}(t)_{\beta(t)} \quad \text{and} \quad \bar{T}(t) = \overset{\frown}{F(\beta)}(t)_{F(\beta(t))}.$$

Omitting the t:

$$T = \tilde{\beta}_{\beta} \quad \text{and} \quad \bar{T} = \overset{\frown}{F(\beta)}_{F(\beta)}.$$

By Theorem III.4.2, $\bar{T} = F_*(T)$

and so $\overset{\frown}{F(\beta)}_{F(\beta)} = F_*(\tilde{\beta}_{\beta})$.

Since F = TC, Theorem III.2.1 tells us that

$$F_*(\tilde{\beta}_{\beta}) = C(\tilde{\beta})_{F(\beta)}$$

and so

$$\overset{\frown}{F(\beta)}_{F(\beta)} = C(\tilde{\beta})_{F(\beta)}$$

and so

$$\overset{\frown}{F(\beta)} = C(\tilde{\beta}).$$

III.5 CONGRUENCE OF CURVES

Introduction

At last we reach the definition of congruence of curves promised in Section III.1 This is the definition suggested in Section III.4 and, as the remarks there show, congruence is in fact an equivalence relation. Results from all the earlier sections of Chapter III are used to prove the main theorem that unit-speed curves are congruent if and only if their curvatures are the same and their torsions differ at most in sign. Using the definitions of Section II.4 we can widen this theorem to include arbitrary-speed regular curves. Finally we generalize the main theorem to non-Frenet frame fields, using the ideas of Section II.6. This last theorem (Theorem III.5.7) is less important now but will be needed in Chapter VI: you are not expected to remember its details.

Some of the proofs in this section are a little long and technical. Try to work through them, but if you get bogged down make sure that you at least understand the statements of the theorems: they are the whole purpose of this chapter.

READ: Section III.5 (pages 116–121).

Comments

(i) *Pages 117–118: proof of Theorem 5.3* Notice the structure of the proof. If F is to carry α to β then certainly $F(\alpha(t)) = \beta(t)$ for all t, so our first requirement is that $F(\alpha(0)) = \beta(0)$. Secondly, we know from Theorem III.4.2 that F_* must take the Frenet frame of α at $\alpha(0)$ to the Frenet of β at $\beta(0)$, reversing the direction of the binormal if $\tau_\alpha = -\tau_\beta$, so we require that F_* have this effect on $T_\alpha(0)$, $N_\alpha(0)$, $B_\alpha(0)$. These two conditions specify F uniquely, so the final stage is to check that $F(\alpha)$ is actually equal to β.

Notice also the beauty and economy of the last stage of the proof. We know that $\bar{T}\cdot T \leqslant 1$ because

$$\bar{T}\cdot T = \|\bar{T}\|\ \|T\| \cos \vartheta,$$

where ϑ is the angle between $\bar{T}$ and T, and $\cos \vartheta = 1$ if and only if ϑ is a multiple of 2π, that is if and only if $\bar{T}$ and T are equal. Thus we can show that $\bar{T} = T$ by showing that $\bar{T}\cdot T$ is always 1. However, because the same remarks apply to $\bar{N}$ and N, $\bar{B}$ and B, we can show that all *three* pairs are equal by the *single* step of showing that

$$f = \bar{T}\cdot T + \bar{N}\cdot N + \bar{B}\cdot B = 3.$$

This is shown by a technique we have noted before, that of proving $f' = 0$ to show f is constant and noting the f(0) has the correct value.

(ii) *Page 119: Corollary 5.5* When O'Neill uses the word "helix" unqualified, he means not a general cylindrical helix, but a *circular* helix, that is a cylindrical helix whose cross-section curve is (part of) a circle (see Exercises II.4.8 and II.4.10), as he remarks in Example I.4.2(2). An equivalent definition of circular helix is as a curve congruent to a reparametrization of the helix

$$t \longmapsto (a \cos t,\ a \sin t,\ bt)$$

given in Example I.4.2(2), and it is this definition O'Neill uses in the proof here.

Supplementary Comments

(i) *Page 117: line −9* F is orientation-preserving because it carries one Frenet frame into another, and Frenet frames are always positively oriented.

(ii) *Page 117: line −2*

$$\begin{aligned} \bar{\kappa} &= \kappa_\alpha, && \text{by Theorem III.4.2,} \\ &= \kappa_\beta, && \text{by the hypotheses of the theorem.} \end{aligned}$$

(iii) *Page 119: Example 5.4* The "a" used here is simply a shorthand for $1/\sqrt{2} = 1/c$: it is not the same "a" as in Example II.3.3.

(iv) *Page 120: first two lines of proof* $\bar{\alpha}$ and $\bar{\beta}$ must be based at the same value of t so that α and β have the same arc-length function s, for if $\bar{\alpha}$ is based at t_1 and $\bar{\beta}$ at t_2 then

$$\begin{aligned} s_\alpha(t) &= \int_{t_1}^{t} v_\alpha(u)du \\ &= \int_{t_1}^{t} v_\beta(u)du, && \text{because } v_\alpha = v_\beta, \\ &= \int_{t_1}^{t_2} v_\beta(u)du + \int_{t_2}^{t} v_\beta(u)du \\ &= s_0 + s_\beta(t), \end{aligned}$$

and the constant s_0 is zero if and only if $t_1 = t_2$.

(v) *Page 121: Theorem 5.7, hypotheses (1) and (2)* These are just natural extensions of the hypotheses in Corollary 5.6, for if $\{E_i\}$ and $\{F_i\}$ are the Frenet frame fields on α, β respectively then condition (1) becomes simply $v_\alpha = v_\beta$ and condition (2) becomes $\kappa_\alpha = \kappa_\beta$, $\tau_\alpha = \tau_\beta$.

(vi) *Page 121: lines 7; 8 of proof* Since F carries the E_i frame at $\alpha(0)$ to the F_i frame at $\beta(0)$, the two left-hand equations are straightforward. For the other two:

$$\begin{aligned} \bar{\alpha}'\cdot\bar{E}_i &= (F(\alpha))'\cdot F_*(E_i), && \text{by definition,} \\ &= F_*(\alpha')\cdot F_*(E_i), && \text{because } F_* \text{ preserves velocities,} \\ &= \alpha'\cdot E_i, && \text{because } F_* \text{ preserves dot products,} \\ &= \beta'\cdot F_i, && \text{by hypothesis (1);} \end{aligned}$$

also

$$\begin{aligned} \bar{E}_i'\cdot\bar{E}_j &= (F_*(E_i))'\cdot F_*(E_j), && \text{by definition,} \\ &= F_*(E_i')\cdot F_*(E_j), && \text{because } F_* \text{ preserves derivatives of vector fields,} \\ &= E_i'\cdot E_j, && \text{because } F_* \text{ preserves dot products,} \\ &= F_i'\cdot F_j, && \text{by hypothesis (2).} \end{aligned}$$

(vii) *Page 121: line 9 of proof* Since $\{\overline{E}_i\}$ is a frame, $\overline{E}_i{}'$ has an orthonormal expansion $\overline{E}_i = \sum_j a_{ij}\overline{E}_j$, where $a_{ij} = \overline{E}_i{}'\cdot\overline{E}_j$ (Theorem II.1.5). Similarly, $F_i{}' = \sum_j b_{ij}F_j$, where $b_{ij} = F_i{}'\cdot F_j$. We have just shown that $F_i{}'\cdot F_j = \overline{E}_i{}'\cdot\overline{E}_j$, so $b_{ij} = a_{ij}$ and we can write $F_i{}' = \sum_j a_{ij}F_j$.

(viii) *Page 121: lines 10, 11 of proof* $\overline{E}_i\cdot\overline{E}_j = \delta_{ij} \Longrightarrow (\overline{E}_i\cdot\overline{E}_j)' = 0$

$$\Longrightarrow \overline{E}_i{}'\cdot\overline{E}_j + \overline{E}_i\cdot\overline{E}_j{}' = 0 \Longrightarrow a_{ij} + a_{ji} = 0.$$

(ix) *Page 121: line 13 of proof*

$$\begin{aligned} f' &= \sum_i (\overline{E}_i{}'\cdot F_i + \overline{E}_i\cdot F_i{}') \\ &= \sum_i \overline{E}_i{}'\cdot F_i + \sum_j \overline{E}_j\cdot F_j{}' \\ &= \sum_{i,j} (a_{ij}\overline{E}_j\cdot F_i + a_{ji}\overline{E}_j\cdot F_i) \\ &= \sum_{i,j} (a_{ij} + a_{ji})\, \overline{E}_j\cdot F_i. \end{aligned}$$

(x) *Page 121: line 14 of proof* For i = 1, 2, 3, since $\overline{E}_i(t)$ and $F_i(t)$ are unit vectors, $\overline{E}_i(t)\cdot F_i(t) \leqslant 1$ with equality if and only if $\overline{E}_i(t)$ and $F_i(t)$ are equal. Thus $f(t) = \sum_i \overline{E}_i(t)\cdot F_i(t) \leqslant 3$ with equality if and only if each $\overline{E}_i(t)$ is equal to the corresponding $F_i(t)$. We have shown that f′ is the zero function, so f itself is a constant function. However, $f(0) = 3$ (because $\overline{E}_i(0) = F_i(0)$ for each i) and so $f(t) = 3$ for all t, that is $\overline{E}_i(t)$ is equal to $F_i(t)$ for all t.

Summary

Definitions

(i)	Congruent curves	Page 116, Definition 5.1
(ii)	Parallel curves	Page 116, line −1

Results

(i)	If $\alpha'(s)$ and $\beta'(s)$ are parallel for each s, α and β are parallel.	Page 117, Lemma 5.2
(ii)	If α and β are parallel and $\alpha(s_0) = \beta(s_0)$ for some s_0, then $\alpha = \beta$.	Page 117, Lemma 5.2
(iii)	Unit-speed curves α, β are congruent if and only if $\kappa_\alpha = \kappa_\beta$ and $\tau_\alpha = \pm\tau_\beta$.	Page 117, Theorem 5.3 and Page 114, Theorem 4.2
(iv)	Arbitrary-speed regular curves α, β are congruent if and only if $v_\alpha = v_\beta$, $\kappa_\alpha = \kappa_\beta$, $\tau_\alpha = \pm\tau_\beta$.	Page 120, Corollary 5.6 and Text, page 26

(v) If α, β are arbitrary curves and $\{E_i\}$, $\{F_i\}$ are frame fields on each such that

$$\alpha'\cdot E_i = \beta'\cdot F_i$$

and $$E_i'\cdot E_j = F_i'\cdot F_j,$$

then α and β are congruent. Page 121, Theorem 5.7

Techniques

(i)	Recognition of congruent curves.	Page 116, Definition 5.1
(ii)	Finding an isometry establishing the congruence of two curves.	Page 117, proof of Theorem 5.3

Exercises

Techniques (i) and (ii)

1. Page 122, Exercise 6.

Theory Exercise

2. Page 122, Exercise 5.

Solutions

1. Page 122, Exercise 6.

$$\alpha(t) = (\sqrt{2}t, t^2, 0)$$
$$\alpha'(t) = (\sqrt{2}, 2t, 0)$$
$$v_\alpha(t) = \sqrt{2}(1 + 2t^2)^{\frac{1}{2}}$$
$$\alpha''(t) = (0, 2, 0)$$

We use the formulas of Theorem II.4.3.

$$\alpha'(t) \times \alpha''(t) = (0, 0, 2\sqrt{2})$$

so

$$\kappa_\alpha(t) = \frac{2\sqrt{2}}{2\sqrt{2}(1 + 2t^2)^{\frac{3}{2}}} = \frac{1}{(1 + 2t^2)^{\frac{3}{2}}}.$$

$$\alpha'''(t) = 0 \quad \text{so} \quad \tau_\alpha(t) = 0.$$

$$\beta(t) = (-t, t, t^2)$$
$$\beta'(t) = (-1, 1, 2t)$$
$$v_\beta(t) = (2 + 4t^2)^{\frac{1}{2}} = \sqrt{2}(1 + 2t^2)^{\frac{1}{2}} = v_\alpha(t)$$
$$\beta''(t) = (0, 0, 2)$$
$$\beta'(t) \times \beta''(t) = (2, 2, 0)$$

so

$$\kappa_\beta(t) = \frac{2\sqrt{2}}{2\sqrt{2}(1 + 2t^2)^{\frac{3}{2}}} = \frac{1}{(1 + 2t^2)^{\frac{3}{2}}} = \kappa_\alpha(t).$$

$$\beta'''(t) = 0 \quad \text{so} \quad \tau_\beta(t) = 0 = \tau_\alpha(t).$$

Thus α and β have the same speed, curvature and torsion and so they are congruent.

$$\alpha'(0) = (\sqrt{2}, 0, 0) \quad \text{so} \quad T_\alpha(0) = (1, 0, 0);$$

$$\alpha'(0) \times \alpha''(0) = (0, 0, 2\sqrt{2}) \quad \text{so} \quad B_\alpha(0) = (0, 0, 1);$$

$$N_\alpha(0) = B_\alpha(0) \times T_\alpha(0) = (0, 1, 0):$$

$$\beta'(0) = (-1, 1, 0) \quad \text{so} \quad T_\beta(0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right);$$

$$\beta'(0) \times \beta''(0) = (2, 2, 0) \quad \text{so} \quad B_\beta(0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right);$$

$$N_\beta(0) = B_\beta(0) \times T_\beta(0) = (0, 0, 1).$$

Since $\tau_\alpha = \tau_\beta$ we want to find the isometry F carrying the α Frenet frame to the β Frenet frame: such an isometry has orthogonal part with matrix

$$C = \underbrace{\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}}_{\text{transpose of attitude matrix of } \beta \text{ frame}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{attitude matrix of } \alpha \text{ frame}} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

Since $\alpha(0) = \beta(0) = \mathbf{0}$, $C(\alpha(0)) = \beta(0)$ and so the translation part of the F is just the identity.

Checking:

$$F(\alpha(t)) = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2}t \\ t^2 \\ 0 \end{pmatrix} = \begin{pmatrix} -t \\ t \\ t^2 \end{pmatrix} = \beta(t).$$

Alternatively, since $\tau_\alpha = -\tau_\beta$ also, we could choose F to take the α Frenet frame to the frame $T_\beta(0)$, $N_\beta(0)$, $-B_\beta(0)$. This gives

$$C = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

and again the translation part of F is the identity.

2. Page 122, Exercise 5.

If α and β are congruent, then $\kappa_\alpha = \kappa_\beta = \kappa$ and $\tau_\alpha = \pm\,\tau_\beta = \tau$. Suppose F is an isometry such that $F(\alpha) = \beta$. Then, as noted in the Additional Text (i), we must have

$$\begin{aligned} F(\alpha(0)) &= \beta(0) \\ F_*(T_\alpha(0)) &= T_\beta(0) \\ F_*(N_\alpha(0)) &= N_\beta(0) \\ F_*(B_\alpha(0)) &= \operatorname{sgn} F\, B_\beta(0) \qquad (*) \end{aligned}$$

by Theorem III.4.2. This theorem also tells us that

$$\tau_\beta = \operatorname{sgn} F\, \tau_\alpha.$$

If $\tau \neq 0$, this equation tells us that $\operatorname{sgn} F = 1$ if $\tau_\alpha = \tau_\beta$, $\operatorname{sgn} F = -1$ if $\tau_\alpha = -\tau_\beta$. Thus the four equations (*) determine the isometry F uniquely, and the proof of Theorem III.5.3 shows that in fact $F(\alpha) = \beta$.

If $\tau = 0$ then there is no constraint on sgn F, so we can have $\operatorname{sgn} F = 1$ or $\operatorname{sgn} F = -1$. In each of these two cases the equations (*) give a unique isometry F. Moreover, since $\tau_\alpha = \tau_\beta$ and $\tau_\alpha = -\tau_\beta$, the proof of Theorem III.5.3 shows that both of these isometries satisfy $F(\alpha) = \beta$.

FURTHER EXERCISES AND SOLUTIONS

Section III.1

Technique Exercises

Page 103, Exercises 5(b), (c).

Solutions

5(b). $(5\sqrt{2}, -5, 4\sqrt{2})$

5(c). $(-2\sqrt{2}, 1, 5\sqrt{2})$.

Section III.2

Technique Exercise

Page 107, Exercise 4(b). (HINT: $(0, 1, 0)$ has the same length as $(1, 0, -1)/\sqrt{2}$.)

Other Recommended Exercise

Page 106, Exercise 3. This proves the result given at the end of page 106 in an alternative way.

Solutions

4(b). There are many possible solutions, each of which is obtained by putting a particular value of ϑ in the following.

$$C = \begin{pmatrix} -\dfrac{\sin\vartheta}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & \dfrac{\cos\vartheta}{\sqrt{2}} \\[2ex] \cos\vartheta & 0 & \sin\vartheta \\[2ex] -\dfrac{\sin\vartheta}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} & \dfrac{\cos\vartheta}{\sqrt{2}} \end{pmatrix}$$

and T is translation through

$$\left(1 + \frac{\sin\vartheta}{2\sqrt{2}} + \frac{1}{\sqrt{2}}, -2 - \frac{\cos\vartheta}{2}, 1 + \frac{\sin\vartheta}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\right).$$

3(a). The translation part of F carries $C(\mathbf{0}) = \mathbf{0}$ to $\mathbf{p}$, so is $T_{\mathbf{p}}$.

The orthogonal part of F carries $U_i(\mathbf{0})$ to $\mathbf{e}_i$, so has the coordinates of the $\mathbf{e}_i$ as its *columns*, so it is ${}^{t}A$.

3(b). The e-frame at p is taken to the natural frame at 0 by $(T_pA^{-1})^{-1} = AT_p^{-1} = T_{A(-p)}A$.

The natural frame at 0 is taken to the f-frame at q by T_qB^{-1}.

$(T_qB^{-1})(T_{A(-p)}A)$ takes the e-frame at p to the f-frame at q and has orthogonal part $B^{-1}A$.

Section III.3

Recommended Exercises

Page 111, Exercise 4. This shows that orthogonal transformations with determinant +1 can be realized as rotations of E^3.

Page 111, Exercise 6. Some more group theory.

Page 111, Exercise 7. This looks at isometries of E^1 and E^2, which are defined in the obvious way.

Solutions

4. Put $e_3 = (p/\|p\|)$. Choose e_1 and e_2 to be orthogonal unit vectors spanning the subspace of E^3 orthogonal to e_3. $C(e_1)$ is also orthogonal to e_3 and has unit length, so

$$C(e_1) = \cos\vartheta\ e_1 + \sin\vartheta\ e_2$$

for some ϑ. If

$$C(e_2) = \sin\vartheta\ e_1 - \cos\vartheta\ e_2,$$

replace ϑ and e_2 by their negatives.

6. For both parts: $\det(AB) = \det A \det B$

$$\det A^{-1} = (\det A)^{-1}$$

$$\det I = +1.$$

7. E^1: $F(x) = \epsilon x + a$

$$E^2\text{: } F = TC,\ C = \begin{pmatrix} \cos\vartheta & \epsilon\sin\vartheta \\ -\sin\vartheta & \epsilon\cos\vartheta \end{pmatrix}.$$

In each case, orientation-preserving if $\epsilon = +1$.

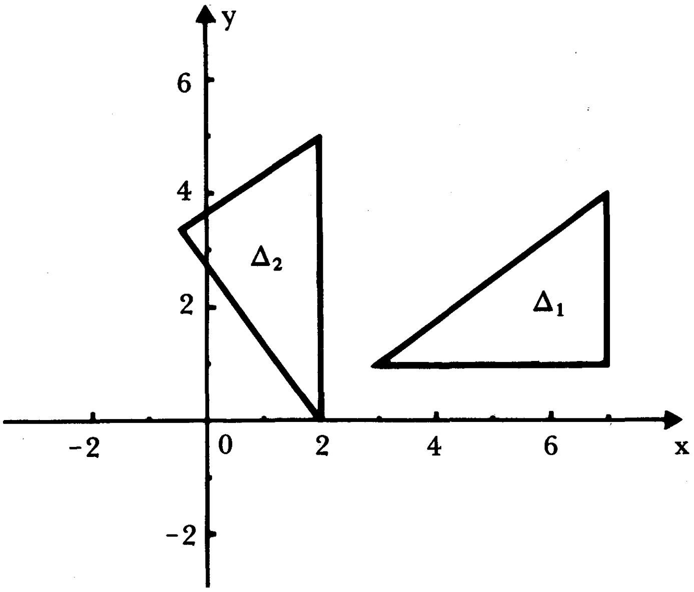
Section III.4

Technique Exercise

Page 116, Exercise 3.

Solution

3.



$$T_{(3,-3)}\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

Section III.5

Technique Exercises

Page 122, Exercises 2,4. (HINT for Exercise 2: The Frenet apparatus is not needed – look for an isometry first.)

Other Recommended Exercises

Page 121, Exercise 1. This gives another characterization of congruence of curves.

Page 122, Exercise 3. This uses Theorem III.5.7 and the connection forms.

Page 122, Exercise 11. This proves a theorem equivalent to Theorem III.4.2 and Theorem III.5.3 for curves in the plane.

Solutions

2. There are eight possible solutions. Each of ϵ, δ, η may be ±1 in the following.

$$\gamma(t) = (\epsilon 3\sqrt{2}t, \delta 3t^2, \eta\sqrt{2}t^3)$$

$$F = \begin{pmatrix} \frac{\epsilon}{\sqrt{2}} & 0 & \frac{\epsilon}{\sqrt{2}} \\ 0 & \delta & 0 \\ -\frac{\eta}{\sqrt{2}} & 0 & \frac{\eta}{\sqrt{2}} \end{pmatrix}$$

4. $\kappa = \frac{1}{4}$, $\tau = -\frac{1}{4}$.

$a = 2$, $b = 2$.

$$F = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

1. If $F = T_pC$, where $C(u_i) = e_i$, then $F(\alpha) = \beta$.

3. The frame field on α is $\{E_i(\alpha)\}$, that on β is $\{E_i(\beta)\}$.

$$\alpha' \cdot E_i(\alpha) = \theta_i(\alpha').$$

$$(E_i(\alpha))' \cdot E_j(\alpha) = (\nabla_{\alpha'} E_i) \cdot E_j(\alpha) = \omega_{ij}(\alpha').$$

11. If: let F be the isometry taking

$\alpha(0)$ to $\beta(0)$

$T_\alpha(0)$ to $T_\beta(0)$

$N_\alpha(0)$ to $\epsilon N_\beta(0)$ where $\kappa_\beta = \epsilon\kappa_\alpha$.

Follow the proof of Theorem III.5.3.

Only if: follow the proof of Theorem III.4.2.

DIFFERENTIAL GEOMETRY

I Calculus on Euclidean Space

II Frame Fields

III Euclidean Geometry

IV Calculus on a Surface

V Shape Operators

VI Geometry of Surfaces in E^3

516.
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DIF

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Set Book

Barrett O'Neill: *Elementary Differential Geometry* (Academic Press, 1966). It is essential to have this book: the course is based on it and will not make sense without it.

Bibliography

The set books for M201, M231 and MST 282 are referred to occasionally; they are useful but not essential. They are:

D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins: *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

E.D. Nering: *Linear Algebra and Matrix Theory* (John Wiley, 1970).

M. Spivak: *Calculus,* paperback edition (W.A. Benjamin/Addison-Wesley, 1973).

R.C. Smith and P. Smith: *Mechanics,* SI Edition (John Wiley, 1972).

Conventions

Before starting work on this text, please read M334 Part Zero. Consult the Errata List and the Stop Press and make any necessary alterations for this chapter in the set book.

Unreferenced pages and sections denote the set book. Otherwise

O'Neill	denotes the set book;
Text	denotes the correspondence text;
KKOP	denotes *An Introduction to Linear Analysis* by D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins;
Nering	denotes *Linear Algebra and Matrix Theory* by E.D. Nering;
Spivak	denotes *Calculus* by M. Spivak;
Smith	denotes *Mechanics* by R.C. Smith and P.Smith.

Reference to Open University Courses in Mathematics take the form:

Unit M100 22, Linear Algebra I

Unit MST 281 10, Taylor Approximation

Unit M201 16, Euclidean Spaces I: Inner Products

Unit M231 2, Functions and Graphs

Unit MST 282 1, Some Basic Tools.

INTRODUCTION

This chapter uses the differential and vector calculus developed in Sections I.1 to II.1.

It introduces the second major theme of this course, *surfaces.* The surfaces in which we shall be interested are subsets of E^3 that locally look like some open region of E^2. You have already met some of these, such as planes, spheres and cylinders.

After discussing several ways of describing a surface we extend the definitions of *tangent vector, differential form* and *mapping* to apply to them.

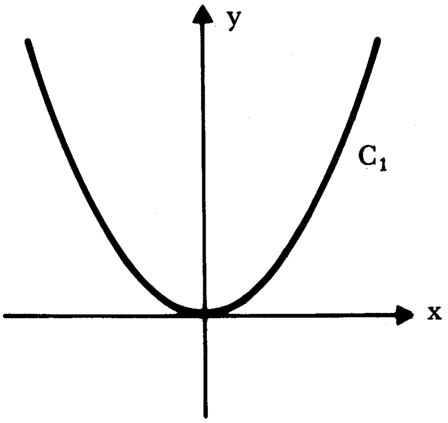
Finally we use differential forms to describe orientability of a surface.

IV.1 SURFACES IN E^3

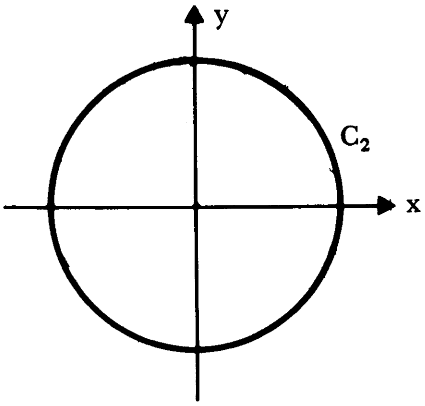
Introduction

This section follows on from Sections I.2, I.4, I.5, I.7 and II.1.

We shall be using implicitly defined curves in the plane. You first met these in Section I.4. Typical examples are:



(a) $C_1 : y - x^2 = 0$



(b) $C_2 : x^2 + y^2 = 1$

In general a curve is defined as a subset C: g = c, where g is a differentiable function and c is a constant. We shall always assume that, for each point (x_0, y_0) in C, either $(\partial g/\partial x)\ (x_0, y_0) \neq 0$ or $(\partial g/\partial y)\ (x_0, y_0) \neq 0$, and that if C has more than one component then we are restricting attention to just one of them. With these restrictions the only curves that we obtain are similar to C_1 and C_2.

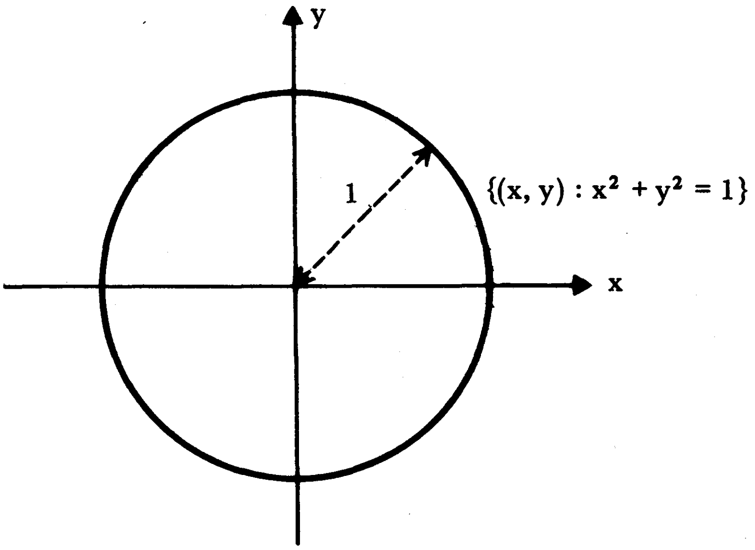
(a) Given a curve like C_1 we can parametrize it by a regular differentiable function α from R to E^2 that is one-to-one and whose inverse map α^{-1} from C_1 to R is continuous. In our example of C_1 such a parametrization is given by $\alpha(t) = (t, t^2)$. This has inverse $\alpha^{-1}(x, y) = x$ for points in C_1.

(b) Given a curve like C_2 we can parametrize it by a periodic regular differentiable function β from R to E^2. We can also demand that if we restrict attention to small enough open subsets of C_2 they will be the one-to-one images of small intervals of R and the inverse functions between these small subsets will be continuous. In our example of C_2 such a parametrization is given by $\beta(t) = (\cos t, \sin t)$. A typical 'local' inverse for this function is given by $\beta^{-1}(x, y) = \cos^{-1} x$ for all those points p in C_2 with $y(p) > 0$.

We shall need the implicit function theorem, a two dimensional form of which was dealt with in an appendix to *Unit M231, Applications of the Derivative,* Section 8.

As an example of an implicitly defined curve in E^2 we can consider again the unit circle

$$C\colon f(x, y) = x^2 + y^2 = 1.$$



We can use the implicit function theorem to describe the local behaviour of this curve. The implicit function theorem states:

Theorem IV.1.A

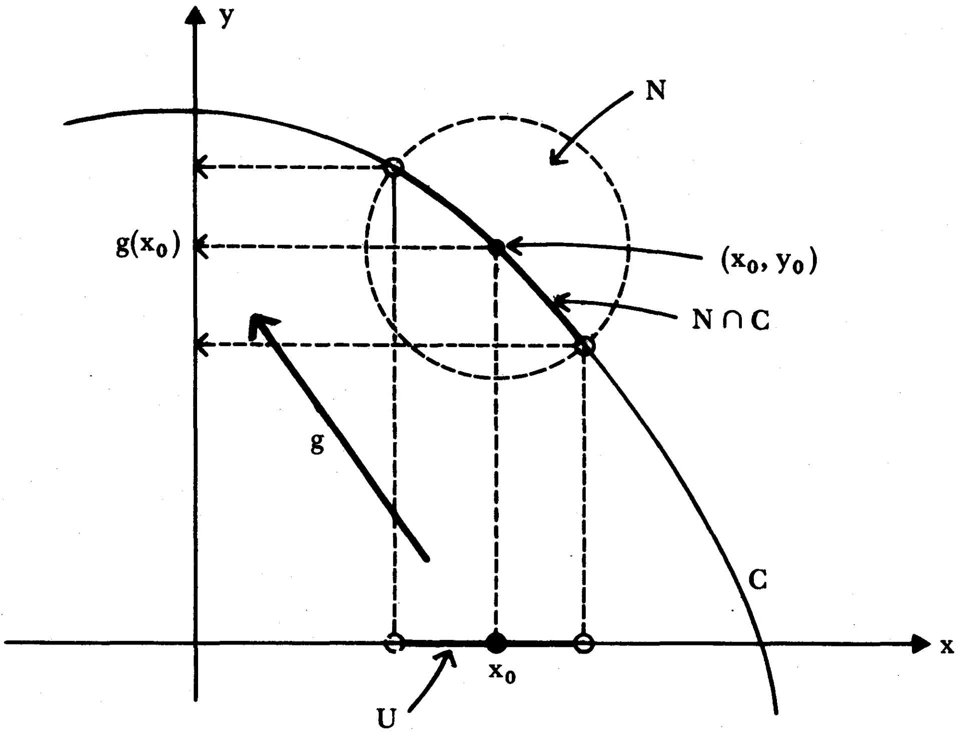
If the point (x_0, y_0) belongs to C and if $(\partial f/\partial y)\,(x_0, y_0) \neq 0$ then there exists:

(i) a neighbourhood U of x_0 in the real line;

(ii) a unique differentiable function $g : U \longrightarrow R$; and

(iii) a neighbourhood N of (x_0, y_0) in the plane; such that

$$N \cap C = \{(t, g(t)) : t \in U\}.$$



That is, locally the curve C can be described as the graph of a mapping $g : U \longrightarrow R$. Similarly we can describe C equally well in terms of y near (x_0, y_0) if $(\partial f/\partial x)\,(x_0, y_0) \neq 0$.

In our example, if $x_0 = \frac{1}{2}$ we can choose for U the open interval $(\frac{1}{4}, \frac{3}{4})$ and then g is the function

$$g : t \longmapsto (1 - t^2)^{\frac{1}{2}}.$$

For a suitably chosen N we have that

$$N \cap C = \{(t, (1 - t^2)^{\frac{1}{2}}) : \tfrac{1}{4} < t < \tfrac{3}{4}\};$$

that is, N ∩ C is the graph of g.

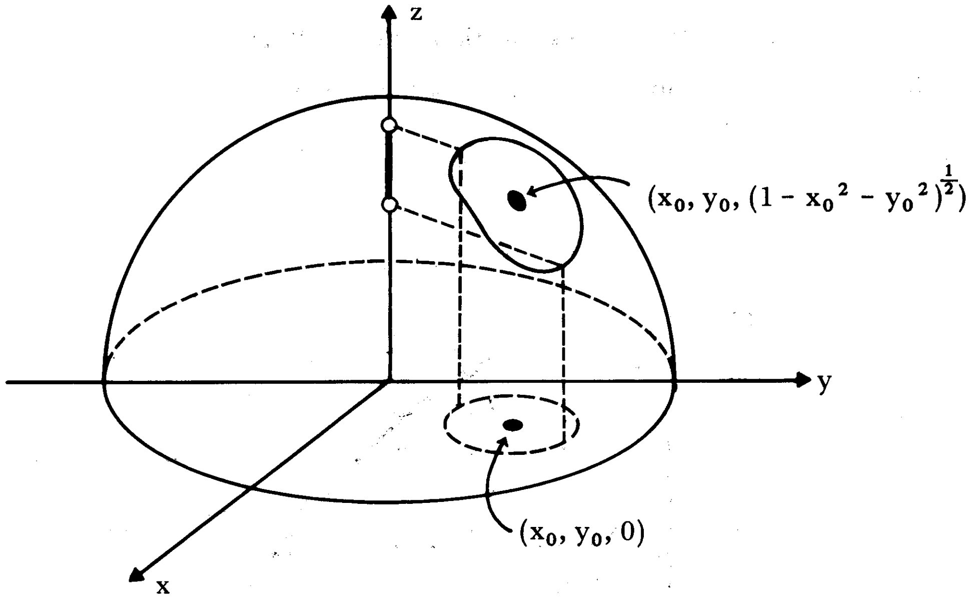
In this section we shall apply an extension of the implicit function theorem to implicitly defined subspaces of E^3. For instance, near a point (x_0, y_0, z_0) on the unit sphere

$$\Sigma : f(x, y, z) = x^2 + y^2 + z^2 = 1$$

for which

$$(\partial f/\partial z)(x_0, y_0, z_0) = 2z_0$$

is nonzero, the sphere can be described as the graph of a function g, where $g(x, y) = z = (1 - x^2 - y^2)^{\frac{1}{2}}$.

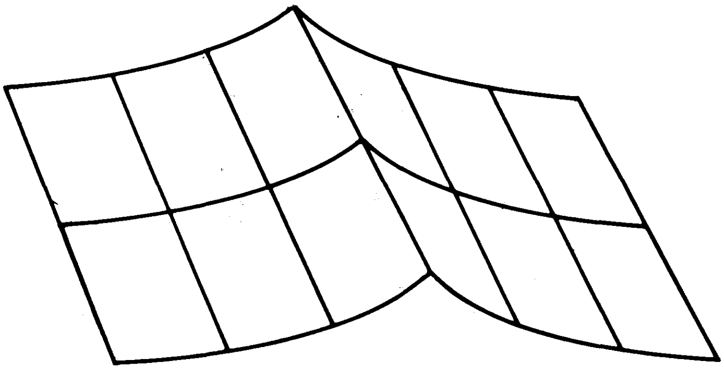


READ: Section IV.1 (pages 124 - 131).

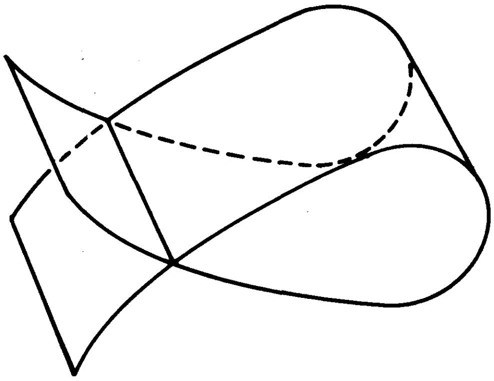
Comments

(i) *The plane E^2* When we use E^2, or some open set D in E^2, as the domain for a patch **x** we write u and v for the natural coordinate functions rather than x and y or x_1 and x_2. Consequently when we want to refer to a variable point in E^2 we write (u, v), though strictly speaking this is the identity function from E^2 to itself. Whichever interpretation you adopt it is unlikely that any ambiguity will arise. When we have to use the partial derivatives of a function f whose domain is in E^2 we write them as $\partial f/\partial u$ and $\partial f/\partial v$.

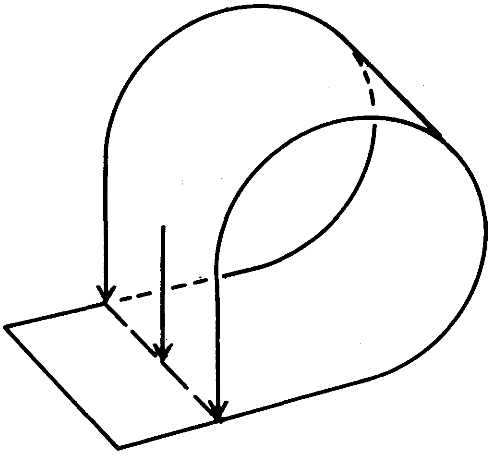
(ii) *Page 124: Proper patches* The conditions in the definition of a proper patch are there to rule out certain pathological possibilities. The condition that the mapping **x** is regular means that its derivative $\mathbf{x}_*$ is one-to-one and hence no nonzero tangent vector is mapped to a zero tangent vector. Since derivatives preserve velocities, each curve with nonzero velocity will be mapped to another curve with nonzero velocity. As we saw in Section I.4 this means that **x** does not introduce any kinks into the patch. The image of a patch cannot look like the following diagram.



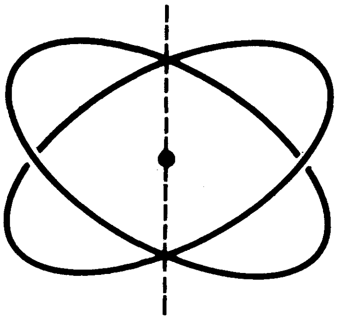
The condition that **x** is one-to-one rules out the possibility that the patch might intersect itself, as in the following diagram.



The condition that $\mathbf{x}^{-1}$ is continuous rules out the possibility that the patch might approach itself arbitrarily closely, as in the following diagram.

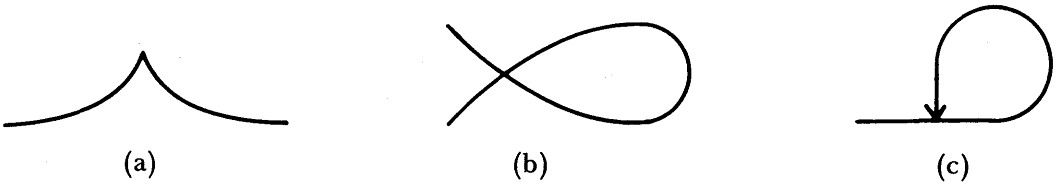


(iii) *Page 125: Definition 1.2* To prove that a subset of E^3 is not a surface we need to find only one point that does not have any neighbourhoods that are like smooth open discs. One way to do this is to look at the boundaries of suitably small neighbourhoods to see if they are like circles. For instance in the second example of Comment (ii) the boundary of a typical small neighbourhood of a point on the line of self-intersection is shown in the following diagram.



We shall assume that our intuition does enable us to distinguish this from a circle.

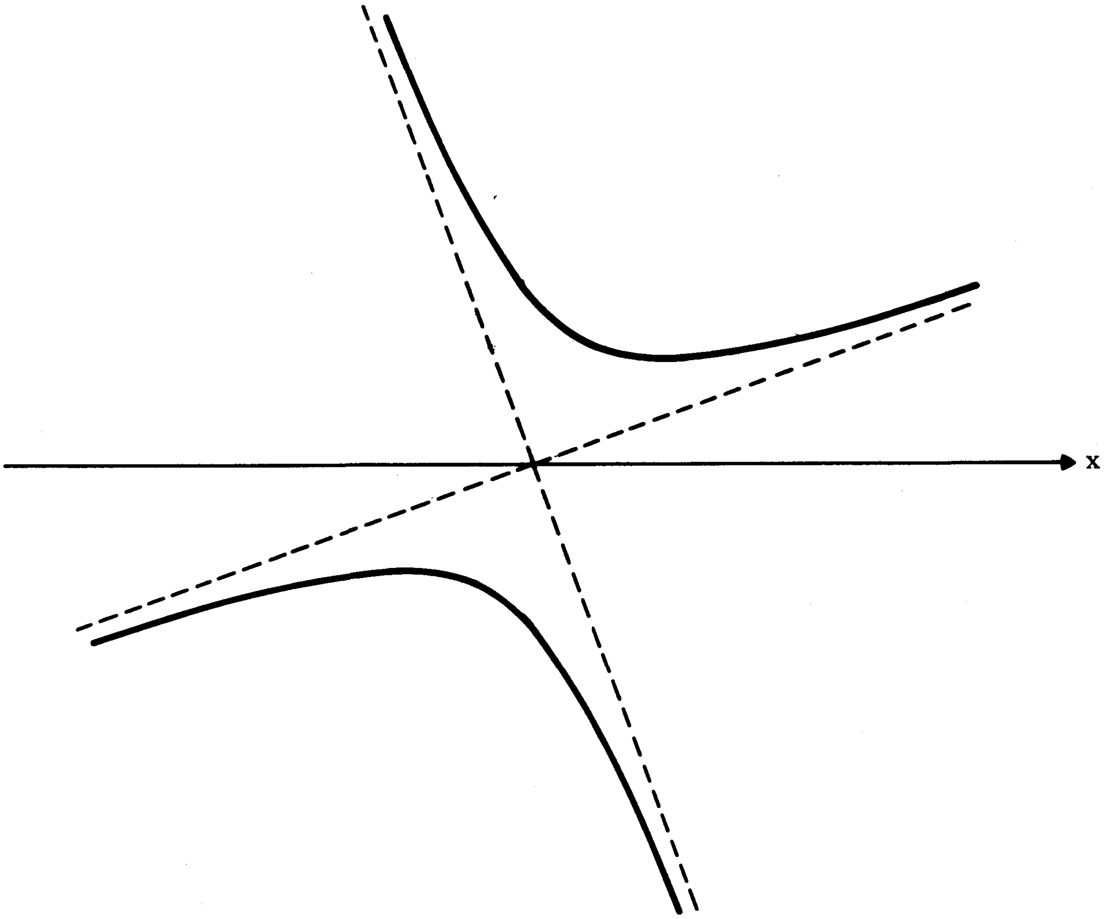
(iv) *Page 129: Curves and surfaces* The curves used in both Example 1.5 and Example 1.6 are defined implicitly. As in Section I.4 this rules out the possibility that they may be of any of the following forms.



(a) (b) (c)

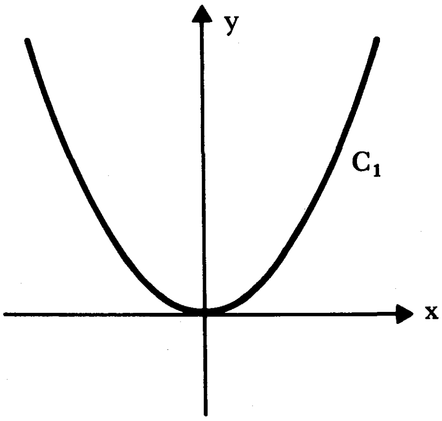
This in turn ensures that the surfaces derived from them, as in Examples 1.5 and 1.6, are not pathological in any of the ways mentioned in Comment (ii). In fact the implicit function theorem discussed in the introduction proves that the above types of curve cannot occur. Curves (b) and (c) cannot be expressed locally as graphs and curve (a) cannot be expressed as the graph of a differentiable function.

When dealing with implicitly defined curves we shall always restrict ourselves to one component at a time. In Example 1.6 this enables us to assume that the curve is in the upper half plane. If we did not have this restriction we would have to consider curves like the following hyperbola.

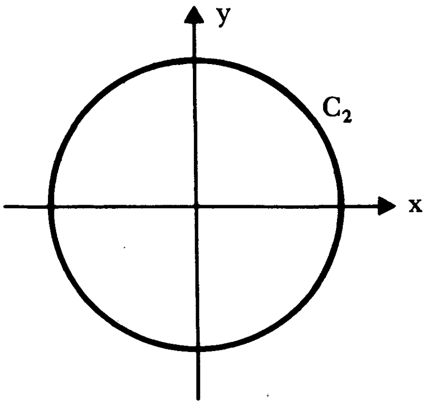


When this is revolved about the x axis the spaces traced out by the two components intersect and the result is not a surface.

As mentioned in the introduction this restriction limits us to two types of curve. The following are typical examples. We can revolve them about any line that does not intersect them.



(a) $C_1 : y - x^2 = 0$

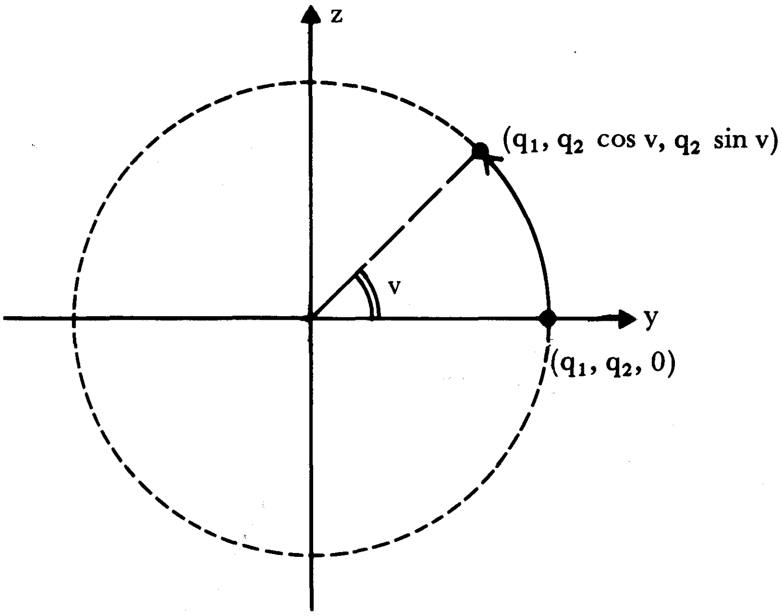


(b) $C_2 : x^2 + y^2 = 1$

For curves like C_1 we shall use only one-to-one parametrizations and with curves like C_2 we shall use only periodic parametrizations. As usual all parametrizations of implicitly defined curves will be regular.

Supplementary Comments

(i) *Page 129: Example 1.6* Rotating about the x axis we keep the x coordinate fixed.



Hence q_2 gives the radius of the circle that the point $(q_1, q_2, 0)$ traces out as it is revolved. When it has been revolved through an angle v its new coordinates are $(q_1, q_2 \cos v, q_2 \sin v)$.

If the point $p = (p_1, p_2, p_3)$ is revolved about the x axis, its x coordinate remains unchanged and its distance from the x axis remain unchanged. This distance is $(p_2^2 + p_3^2)^{\frac{1}{2}}$. Hence when it has been swung round into the upper half of the xy plane it reaches the point $\overline{p} = (p_1, (p_2^2 + p_3^2)^{\frac{1}{2}}, 0)$. The point p is on the surface of revolution if and only if this point is on the original curve. So (p_1, p_2, p_3) belongs to M if and only if $f(p_1, (p_2^2 + p_3^2)^{\frac{1}{2}}) = c$. That is, M is given by $M : g = c$ where $g(x, y, z) = f(x, (y^2 + z^2)^{\frac{1}{2}})$.

How do we use the chain rule to prove that dg is never zero? Firstly, since g is a mapping from E^3 to R

$$dg(v) = v[g] = \frac{d}{dt}(g(p + tv))\Big|_{t=0} = g_*(v)$$

for any tangent vector v to E^3 at p. Similarly for any tangent vector w to E^2 at q then

$$df(w) = f_*(w),$$

where f is a function from E^2 to R and the differential df is defined by analogy with the definition for mappings from E^3 to R. Now g is the composite mapping $f \circ F$ where

$$F(p_1, p_2, p_3) = (F_1(p_1, p_2, p_3), F_2(p_1, p_2, p_3))$$
$$= (p_1, (p_2^2 + p_3^2)^{\frac{1}{2}}).$$

Hence by the chain rule

$$dg(v) = g_*(v) = f_*(F_*(v)) = df(F_*(v)).$$

Now the Jacobian matrix for F is

$$F_* = \begin{pmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\ & & \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ & & \\ 0 & \dfrac{y}{(y^2+z^2)^{\frac{1}{2}}} & \dfrac{z}{(y^2+z^2)^{\frac{1}{2}}} \end{pmatrix}$$

Hence at points of M the mapping F_* is onto, has rank 2, since either y or z is nonzero as M does not intersect the x axis.

Assume dg is zero at p on M; then $dg(v) = 0$ for all vectors v tangent to M at p. Now if w is tangent vector to E^2 at F(p): then, since F_* is onto, $w = F_*(v)$ for some tangent vector v to E^3 at p. Hence

$$df(w) = df(F_*(v)) = dg(v) = 0,$$

by assumption, and so df is zero at F(p). Now for mappings from E^2 to R the analogue of Corollary I.5.5 gives

$$df = \frac{\partial f}{\partial u}\,du + \frac{\partial f}{\partial v}\,dv.$$

Hence $\partial f/\partial u$ and $\partial f/\partial v$ are both zero at F(p). But

$$F(p) = (p_1, (p_2{}^2 + p_3{}^2)^{\frac{1}{2}})$$

is a point of the profile curve C and so by the definition of an implicitly defined curve $\partial f/\partial u$ and $\partial f/\partial v$ cannot both be zero. This contradiction tells us that our original assumption that dg was zero at p is false; dg is never zero on M.

Summary

Notation

x	Page 124, Definition 1.1
M	Page 125, Definition 1.2
Σ	Page 125, line -6
M : z = f(x, y)	Page 127, Example 1.3
M : g = c	Page 127, line -11

Definitions

(i)	Coordinate patch $x : D \longrightarrow E^3$	Page 124, Definition 1.1
(ii)	Proper patch $x : D \longrightarrow E^3$	Page 124, line -2
(iii)	Surface M	Page 125, Definition 1.2
(iv)	Monge patch x(u, v) = (u, v, f(u, v))	Page 127, line 8
(v)	Simple surface M = x(D)	Page 127, line 11
(vi)	Graph of a function M : z = f(x, y)	Page 127, Example 1.3
(vii)	Implicitly defined surface M : g = c	Page 127, line -11
(viii)	Cylinder	Page 129, Example 1.5
(ix)	Surface of revolution	Page 129, Example 1.6
(x)	Parallel	Page 130, line 8
(xi)	Meridian	Page 130, line 9

Examples

(i)	Unit sphere Σ	Page 125, line −3
(ii)	Hemispherical patches on the sphere, $x(u, v) = (u, v, \pm(1-u^2-v^2)^{\frac{1}{2}})$	Page 126, line 7
(iii)	Sphere	Page 128, line −2
(iv)	Circular cylinder, $M : x^2 + y^2 = r^2$ in E^3	Page 129, line −13

Results

(i)	Graphs are surfaces.	Page 127, Example 1.3
(ii)	An implicitly defined subset $M : g = c$ is a surface if the differential dg is never zero at any point of M.	Page 127, Theorem 1.4
(iii)	Cylinders are surfaces.	Page 129, Example 1.5
(iv)	Surfaces of revolution are surfaces.	Page 129, Example 1.6

Techniques

(i)	Recognizing a (proper) patch.	Page 124, Definition 1.1 (and Page 124, line −2)
(ii)	Determining whether subsets of E^3 are surfaces by looking for proper patches.	Page 125, Definition 1.2
(iii)	Recognizing that a subset of E^3 is a surface since it is a graph, a cylinder or a surface of revolution.	Page 127, Example 1.3, Page 129, Example 1.5, Page 129, Example 1.6
(iv)	Determining whether an implicitly defined subset of E^3, $M : g = c$, is a surface by evaluating the differential dg.	Page 127, Theorem 1.4
(v)	Sketching surfaces.	

Exercises

Technique (i)

1. Page 132, Exercise 4.

Techniques (ii) and (iii)

2. Page 131, Exercise 1. Deleting as few points as possible from each subset construct a surface. Explain why it is a surface.

Technique (iv)

3. Page 132, Exercise 5.

Technique (v)

4. Page 132, Exercise 6.

Solutions

1. Page 132, Exercise 4.

The Jacobian matrix for the mapping $\mathbf{x} = (x_1, x_2, x_3) : E^2 \longrightarrow E^3$ is

$$\begin{pmatrix} \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_1}{\partial v} \\ \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_2}{\partial v} \\ \dfrac{\partial x_3}{\partial u} & \dfrac{\partial x_3}{\partial v} \end{pmatrix}$$

(a) The Jacobian matrix is

$$\begin{pmatrix} 1 & 0 \\ v & u \\ 0 & 1 \end{pmatrix}$$

This always has rank 2 and so **x** is regular.

Now if

$$\mathbf{x}(u, v) = \mathbf{x}(u', v')$$

then

$$(u, uv, v) = (u', u'v', v')$$

and so

$$u = u', v = v'.$$

Hence **x** is one-to-one.

Since **x** is one-to-one and regular it is a patch.

(b) The Jacobian matrix is

$$\begin{pmatrix} 2u & 0 \\ 3u^2 & 0 \\ 0 & 1 \end{pmatrix}$$

When $u = 0$ this matrix has rank 1. Hence **x** is not regular and so is not a patch.

(c) The Jacobian matrix is

$$\begin{pmatrix} 1 & 0 \\ 2u & 0 \\ 0 & 1+3v^2 \end{pmatrix}$$

and since $v^2 \geqslant 0$ it has rank 2. Hence **x** is regular.

Suppose $\mathbf{x}(u, v) = \mathbf{x}(u', v')$.

Then

$$(u, u^2, v + v^3) = (u', u'^2, v' + v'^3).$$

Hence

$$u = u' \quad \text{and} \quad (v - v') + (v^3 - v'^3) = 0.$$

It follows that

$$(v - v')\,(1 + (v^2 + vv' + v'^2)) = 0$$

and hence

$$(v - v')\,(1 + (v + \frac{v'}{2})^2 + \frac{3}{4}v'^2) = 0.$$

Since $1 + (v + \frac{v'}{2})^2 + \frac{3}{4}v'^2 > 0$ it follows that $v = v'$ also and hence **x** is one-to-one. Since **x** is one-to-one and regular it is a patch.

(d) If $\mathbf{x}(u, v) = \mathbf{x}(u', v')$ then

$$\cos 2\pi u = \cos 2\pi u', \quad \sin 2\pi u = \sin 2\pi u' \quad \text{and} \quad v = v'.$$

These conditions are satisfied whenever $u = u' + n$, for any integer n, and $v = v'$. Hence the mapping is not one-to-one and **x** is not a patch.

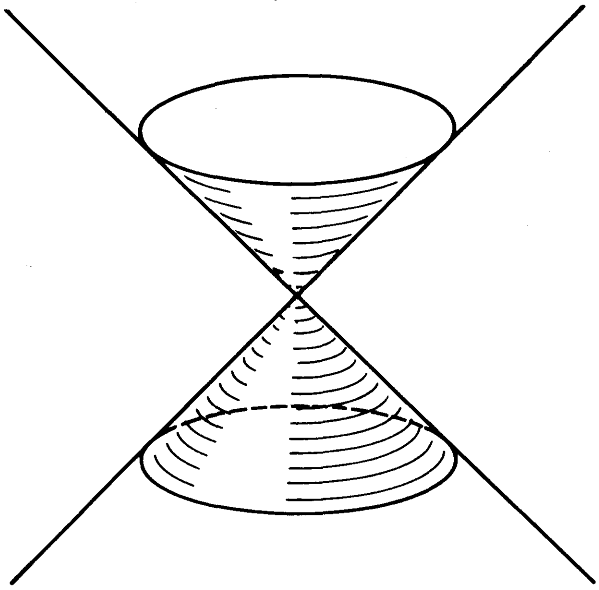
2. Page 131, Exercise 1.

(a) We can write M as $M : g = 0$, where $g(x, y, z) = z^2 - x^2 - y^2$. By Corollary I.5.5 the differential of g is

$$dg = -2x\,dx - 2y\,dy + 2z\,dz$$

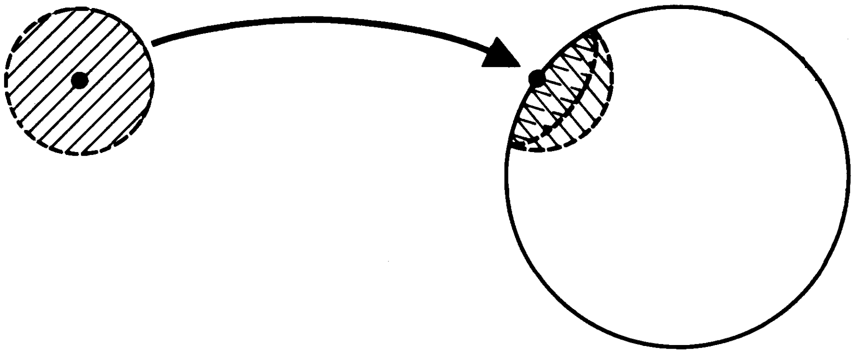
and this is zero only at the origin. Hence, by Theorem IV.1.4, the cone minus its apex is a surface.

Since the boundary of a small neighbourhood of the origin is two circles we cannot include the origin if we require a surface.

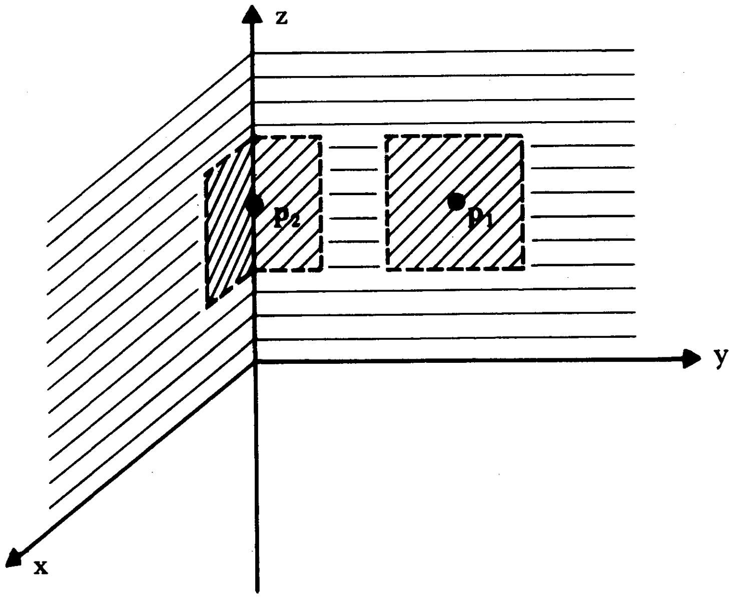


(b) Points not on the boundary of the closed disc have the same small neighbourhoods as if they were considered as points of the plane $P : z = 0$. Since the differential dz is nonzero it follows from Theorem IV.1.4 that the disc minus its boundary is a surface.

If we try to map a small open disc continuously to the closed disc in such a way that the centre of the open disc is mapped to a point on the boundary of the closed disc we find that we have to fold it over. Hence the map cannot be one-to-one and so the boundary points cannot be included in the surface.



(c)



The region $M : xy = 0, x \geqslant 0, y \geqslant 0$ can be written as the union of two half planes $M_1 : x = 0, y \geqslant 0$ and $M_2 : y = 0, x \geqslant 0$. For any point p_1 in the interior of either of these planes it is easy to find a rectangular coordinate patch. Thus the subset of M obtained by deleting the z axis is a surface. However, for any point p_2 on the z axis, the intersection of these two half planes, it is impossible to find such a patch. Any neighbourhood of p_2 has a right angle bend in it and so no mapping $x : D \longrightarrow E^3$ onto a neighbourhood of p_2 will be differentiable and regular at p_2.

3. Page 132, Exercise 5.

(a) By Theorem IV.1.4, M : $(x^2 + y^2)^2 + 3z^2 = 1$ is a surface if dg is never zero on M, where $g(x, y, z) = (x^2 + y^2)^2 + 3z^2$. Now

$$dg = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz, \quad \text{by Corollary 1.5.5,}$$

and hence

$$\begin{aligned} dg &= 2(x^2 + y^2)\, 2x\, dx + 2(x^2 + y^2)\, 2y\, dy + 6z\, dz \\ &= 4(x^2 + y^2)\,(x\, dx + y\, dy) + 6z\, dz. \end{aligned}$$

Hence dg is zero only at the origin and since this is not a point of M it follows that M is a surface.

(b) By Theorem IV.1.4, M : $z(z - 2) + xy = c$ is a surface if dg is never zero on M, where $g(x, y, z) = z(z - 2) + xy$.

Now

$$dg = 2(z - 1)\, dz + x\, dy + y\, dx$$

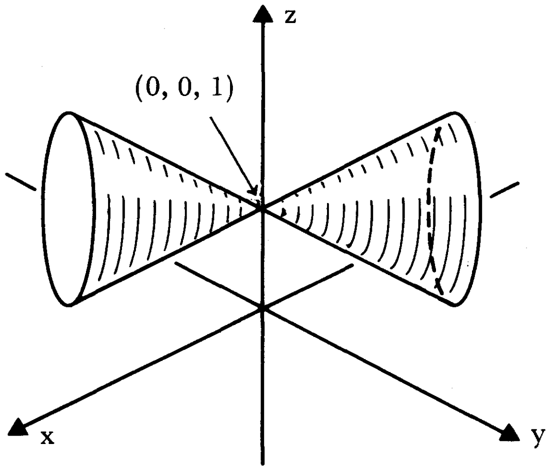
and hence dg is zero only at the point (0, 0, 1).

The point (0, 0, 1) belongs to M if and only if

$$\begin{aligned} &(z(z - 2) + xy)\,(0, 0, 1) = c \\ &1(1 - 2) + 0.0 = c \\ &\qquad c = -1. \end{aligned}$$

Hence M is a surface except possibly when $c = -1$.

When $c = -1$, the set M is given by M : $z(z - 2) + xy = -1$ or, on rearranging, M : $(z - 1)^2 + xy = 0$. A sketch of the set M near the point (0, 0, 1) shows that it is a 'double' cone. In Solution 2, above, we saw that this is not a surface.



4. Page 132, Exercise 6.

M : $z = y^3 - 3yx^2$

The intersection of M with the xy plane is given by

S : $0 = y^3 - 3yx^2,\ z = 0$.

Now if

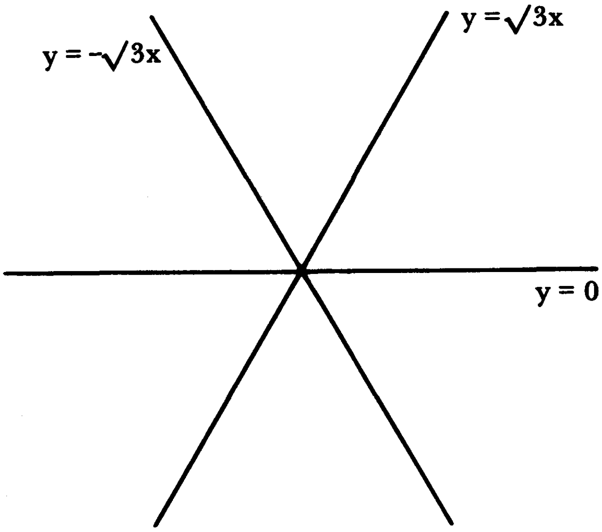
$$0 = y^3 - 3yx^2$$

then

$$\begin{aligned} 0 &= (y^2 - 3x^2)y \\ &= (y - \sqrt{3}x)(y + \sqrt{3}x)y. \end{aligned}$$

The intersection consists of three lines in the xy plane. The lines are $y = 0$, $y - \sqrt{3}x = 0$ and $y + \sqrt{3}x = 0$.

These lines make angles of $\frac{\pi}{3}$ with each other.



As a corollary to the intermediate value theorem, we can deduce that throughout each of the six resulting regions of the plane the function $f(x, y) = y^3 - 3yx^2$ is always positive or always negative. All we need to do is to evaluate f at one point of each region. The values are alternately positive and negative. Hence the surface M is alternately above and below the plane. This surface is illustrated in Figure 5.22 on page 205.

IV.2 PATCH COMPUTATIONS

Introduction

This section follows on directly from Section IV.1 and assumes familiarity with curves, dealt with in Section I.4, and the dot and cross product, dealt with in Section II.1. It is also useful to bear in mind the definition of the derivative of a mapping, given in Section I.7.

Choosing patches enables us to describe surfaces in terms of regions of the plane E^2. In this section we begin to see how we can use the geometry of E^2 to study surfaces.

This section contains many important examples, which will be used again and again throughout the remainder of the course.

READ: Section IV.2 (pages 133 - 140).

Comment

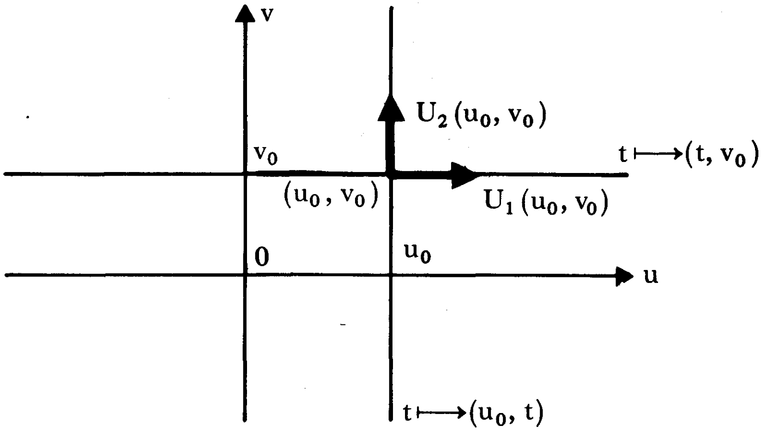
(i) *Patches* Through each point (u_0, v_0) in the plane E^2 there are two curves

$$t \longmapsto (t, v_0)$$

and $\quad t \longmapsto (u_0, t),$

that are parallel to the coordinate axes.

These are unit-speed curves and their velocities at the point (u_0, v_0) are $U_1(u_0, v_0)$ and $U_2(u_0, v_0)$.



The images of these curves under the mapping x are the u- and v-parameter curves, $v = v_0$ and $u = u_0$ respectively. These parameter curves depend on the choice of coordinate patch x.

Since regular mappings preserve velocity vectors, the partial velocities (the velocities of the parameter curves) are given by

$$x_u(u_0, v_0) = x_*(U_1(u_0, v_0))$$

and

$$x_v(u_0, v_0) = x_*(U_2(u_0, v_0)).$$

Hence $\mathbf{x}_u$ and $\mathbf{x}_v$ are the composite mappings $\mathbf{x}_*(U_1)$ and $\mathbf{x}_*(U_2)$. To remind ourselves that $\mathbf{x}_u$ and $\mathbf{x}_v$ are functions on E^2 we sometimes write them as $\mathbf{x}_u(u, v)$ and $\mathbf{x}_v(u, v)$. To calculate these explictly we look at the parameter curves. The u-parameter curve $v = v_0$ is given by

$$\alpha : t \longmapsto \mathbf{x}(t, v_0) = (x_1(t, v_0), x_2(t, v_0), x_3(t, v_0)).$$

Hence

$$\mathbf{x}_u(u_0, v_0) = \frac{d\alpha}{dt}(u_0) \quad \text{at } \mathbf{x}(u_0, v_0)$$

$$= \left(\frac{\partial x_1}{\partial u}(u_0, v_0), \frac{\partial x_2}{\partial u}(u_0, v_0), \frac{\partial x_3}{\partial u}(u_0, v_0)\right) \quad \text{at } \mathbf{x}(u_0, v_0).$$

Since $\mathbf{x}$ is regular, the derivate $\mathbf{x}_*$ is one-to-one and hence it carries linearly independent vectors into linearly independent vectors. The tangent vectors $U_1(u_0, v_0)$ and $U_2(u_0, v_0)$ are linearly independent in E^2 and so $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$ are linearly independent in E^3. In Section II.1 we saw that this is equivalent to the condition

$$\mathbf{x}_u(u_0, v_0) \times \mathbf{x}_v(u_0, v_0) \neq \mathbf{0}.$$

That is, $\mathbf{x}_u \times \mathbf{x}_v$ is never zero.

In fact this condition that $\mathbf{x}_u \times \mathbf{x}_v$ is never zero is enough to ensure that $\mathbf{x}$ is regular. The condition ensures that the images of $U_1(u_0, v_0)$ and $U_2(u_0, v_0)$ under $\mathbf{x}_*$ are linearly independent and this implies that $\mathbf{x}_*$ is one-to-one at each point.

When O'Neill comes to calculate $\mathbf{x}_u \times \mathbf{x}_v$ he obtains the equation

$$\mathbf{x}_u \times \mathbf{x}_v = \begin{vmatrix} U_1 & U_2 & U_3 \\ \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_3}{\partial u} \\ \dfrac{\partial x_1}{\partial v} & \dfrac{\partial x_2}{\partial v} & \dfrac{\partial x_3}{\partial v} \end{vmatrix}$$

Here U_1, U_2, U_3 are vector valued functions with domain E^3 while all the other functions have domain E^2 and hence the equation is not technically correct. However it is very convenient to use such expressions as long as you are always aware of how they could be made rigorous. To make it rigorous we could replace U_1, U_2, U_3 by the composite functions $U_1 \circ \mathbf{x}, U_2 \circ \mathbf{x}$ and $U_3 \circ \mathbf{x}$, or state that if the functions with the domain E^2 are evaluated at the point (u_0, v_0) then those with domain E^3 are evaluated at the point $\mathbf{p} = \mathbf{x}(u_0, v_0)$.

Supplementary Comments

(i) *Page 136: line -1*

$$r^{-2}\mathbf{x_u} \times \mathbf{x_v} = \begin{vmatrix} U_1 & U_2 & U_3 \\ -\cos v \sin u & \cos v \cos u & 0 \\ -\sin v \cos u & -\sin v \sin u & \cos v \end{vmatrix}$$

$$= \cos^2 v \cos u\, U_1 - \cos v\,(-\cos v \sin u)\, U_2$$

$$+((-\cos v \sin u)(-\sin v \sin u) - (-\sin v \cos u)(\cos v \cos u))\, U_3$$

$$= \cos^2 v \cos u\, U_1 + \cos^2 v \sin u\, U_2 + \cos v \sin v\, U_3$$

since $\cos^2 u + \sin^2 u = 1$.

(ii) *Page 139: line 2,* $\mathbf{x_u}$ *and* $\mathbf{x_v}$ *are linearly independent*

Here

$$\mathbf{x_u}(u, v) = (g'(u), h'(u) \cos v, h'(u) \sin v)$$
$$\mathbf{x_v}(u, v) = (0, -h(u) \sin v, h(u) \cos v)$$

and hence

$$\mathbf{x_u}(u, v) \times \mathbf{x_v}(u, v) = (h(u)\, h'(u), -h(u)g'(u) \cos v, -h(u)g'(u) \sin v).$$

If $\mathbf{x_u}(u, v)$ and $\mathbf{x_v}(u, v)$ are linearly dependent at $\mathbf{x}(u_0, v_0)$ then

$$\mathbf{x_u}(u_0, v_0) \times \mathbf{x_v}(u_0, v_0) = 0.$$

But since $h(u_0) > 0$ and $\cos v_0$ and $\sin v_0$ cannot simultaneously be zero, we would have $h'(u_0) = g'(u_0) = 0$. This is impossible since by the definition (on page 21) a parametrization is a regular curve. Hence $\mathbf{x_u}(u, v) \times \mathbf{x_v}(u, v)$ is never zero and the partial velocities $\mathbf{x_u}(u, v)$ and $\mathbf{x_v}(u, v)$ are linearly independent.

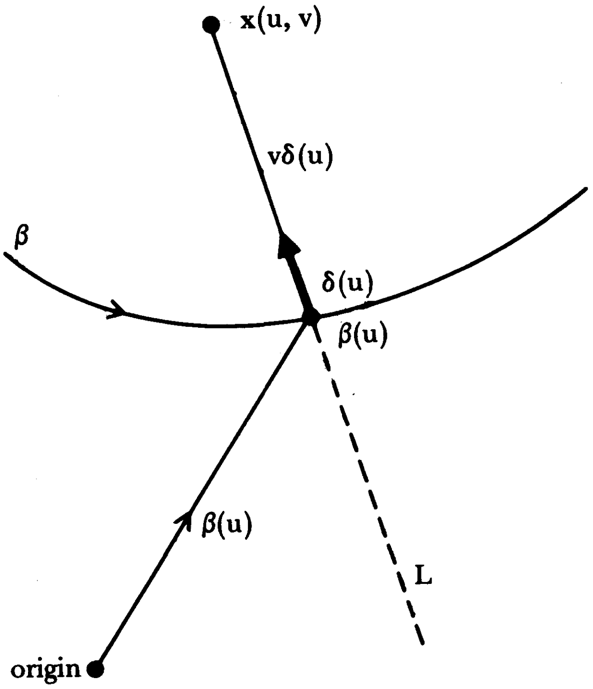
Additional Text

There are two important classes of surfaces defined in the Exercises at the end of Section IV.2.

(i) *Ruled Surfaces*

READ the definition on page 140 between Exercises 3 and 4 and look at the examples in Exercises 5 and 6.

Comment $\beta(u)$ gives the position along the base curve, $\delta(u)$ gives the 'direction' of the line L and v gives the 'distance' along this line from the base curve.



δ(u) need not be the same length and direction for each u.

(ii) *Quadric surfaces*

READ the definition on page 142, between Exercises 9 and 10, and browse through the examples in Exercises 10 to 11.

Summary

Notation

$x_u(u_0,v_0)$ and $x_v(u_0,v_0)$	Page 134, Definition 2.1
x_u and x_v	Page 134, line 12

Definitions

(i)	u- and v- parameter curves, $v = v_0$ and $u = u_0$	Page 133, line −9
(ii)	Partial velocities, $x_u(u_0,v_0)$ and $x_v(u_0,v_0)$	Page 134, Definition 2.1
(iii)	Parametrization	Page 135, Definition 2.3
(iv)	Cross-sectional curve of a cylinder	Page 138, line 9
(v)	Rulings of a cylinder	Page 138, line 11
(vi)	Ruled surface	Page 140, lines −16 to −8
(vii)	Rulings	
(viii)	Ruled form	
(ix)	Base curve	
(x)	Director curve	
(xi)	Quadric surface	Page 142, line 16

Examples

(i)	The geographical patch on the sphere: $\mathbf{x}(u, v) = (r \cos v \cos u, r \cos v \sin u, r \sin v)$, where $u \in (-\pi, \pi), v \in (-\frac{\pi}{2}, \frac{\pi}{2})$	Page 135, line 3
(ii)	Parametrization of a cylinder: $\mathbf{x}(u, v) = (\alpha_1(u), \alpha_2(u), v)$	Page 137, Example 2.4
(iii)	Parametrization of a surface of revolution: $\mathbf{x}(u, v) = (g(u), h(u) \cos v, h(u) \sin v)$	Page 138, Example 2.5
(iv)	Parametrization of a torus of revolution: $\mathbf{x}(u, v) = ((R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u)$	Page 139, Example 2.6
(v)	Parametrization of a ruled surface in ruled form: $\mathbf{x}(u, v) = \beta(u) + v\delta(u)$	Page 140, line -12
(vi)	Helicoid: $\mathbf{x}(u, v) = (u \cos v, u \sin v, bv)$	Page 141, Exercise 7

Results

(i)	$\mathbf{x} : D \longrightarrow E^3$ is regular if and only if $\mathbf{x}_u \times \mathbf{x}_v$ is never zero.	Page 136, line 6
(ii)	The partial velocities are the images of the standard basis vectors under the derivative mapping $\mathbf{x}_*$: $\mathbf{x}_*(U_1) = \mathbf{x}_u, \quad \mathbf{x}_*(U_2) = \mathbf{x}_v$.	Text, page 20

Techniques

(i)	Description of the parameter curves.	Page 133, line -9
(ii)	Evaluation of partial velocities.	Page 134, line -7
(iii)	Testing regularity of a parametrization.	Page 136, line 6
(iv)	Use of standard parametrization of cylinders and surfaces of revolution.	Page 137, Example 2.4, Page 138, Example 2.5
(v)	Finding the ruled form of a surface, in certain simple examples.	Page 140, line -16
(vi)	Familiarity with the shapes and equations of quadric surfaces.	Page 142, line 16

Exercises

Techniques (i), (ii) and (iii)

1. Page 141, Exercise 7.

Technique (iii)

2. Page 140, Exercise 2.

Technique (iv)

3. Page 141, Exercise 6.
4. Page 140, Exercise 1.

Technique (v)

5. Page 140, Exercise 4.

Technique (vi)

6. Page 142, Exercises 10 and 11. Only sketch the surfaces.
7. Page 142, Exercise 12(a) and (b).

Solutions

1. Page 141, Exercise 7.

(a) To check that **x** is a patch we must show that it is one-to-one and regular. If $\mathbf{x}(u, v) = \mathbf{x}(u', v')$ then

$$u \cos v = u' \cos v', \quad u \sin v = u' \sin v' \quad \text{and} \quad v = v'.$$

Since $v = v'$ and the cosine and sine functions cannot be simultaneously zero it follows that $u = u'$. Hence **x** is one-to-one.

Now

$$\mathbf{x}_u \times \mathbf{x}_v = \begin{vmatrix} U_1 & U_2 & U_3 \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & b \end{vmatrix}$$

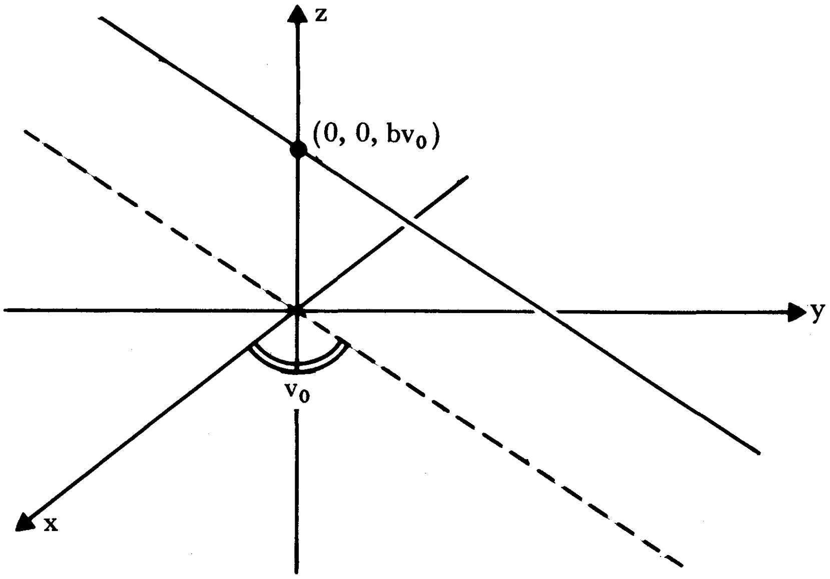
$$= (b \sin v, -b \cos v, u).$$

Hence $\|\mathbf{x}_u \times \mathbf{x}_v\|^2 = b^2 + u^2 > 0$, and so $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$. Thus **x** is regular and therefore a patch.

(b) The u-parameter curve $v = v_0$ is

$$\begin{aligned} t \longmapsto \mathbf{x}(t, v_0) &= (t \cos v_0, t \sin v_0, bv_0) \\ &= (0, 0, bv_0) + t(\cos v_0, \sin v_0, 0), \end{aligned}$$

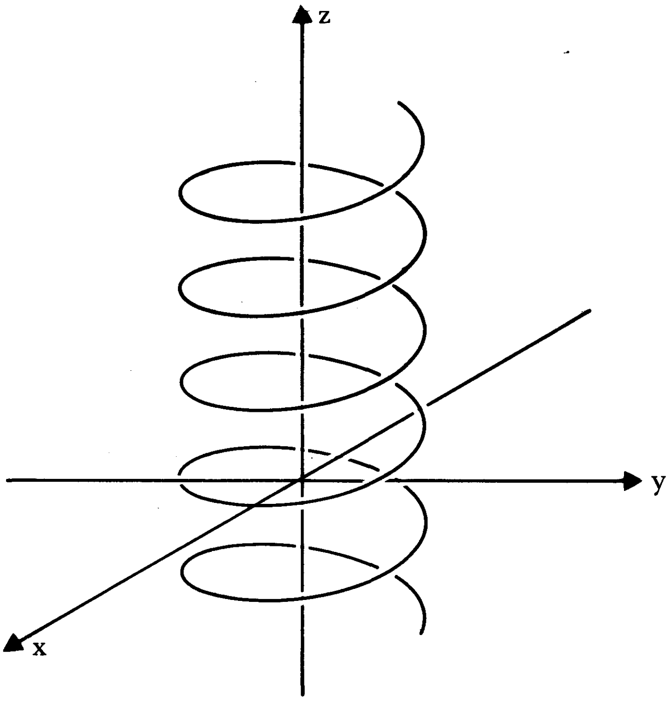
which is a straight line parallel to the line in the xy plane making an angle v_0 with the x axis.



The v–parameter curve $u = u_0$ is

$$t \longmapsto \mathbf{x}(u_0, t) = (u_0 \cos t, u_0 \sin t, bt).$$

As t increases by 2π the curve goes once round the z axis, in a circle of radius u_0, and rises $2\pi b$. The v–parameter curves are helices about the z axis.



(c) In order to express the helicoid in implicit form we must find some function g and constant c such that $g(\mathbf{p}) = c$ if and only if $\mathbf{p} = \mathbf{x}(u_0, v_0)$ for some (u_0, v_0).

If $\mathbf{p} = \mathbf{x}(u_0, v_0)$ then

$$p_3 = bv_0, \quad v_0 = p_3/b$$

and

$$p_1 = u_0 \cos(p_3/b), \quad p_2 = u_0 \sin(p_3/b).$$

Hence for each point p of the helicoid

$$p_1 \sin (p_3/b) = p_2 \cos (p_3/b)$$

so if a point belongs to the helicoid it also belongs to the implicitly defined surface

$$M : x \sin(z/b) - y \cos(z/b) = 0.$$

We need to show conversely that any point of M belongs to the helicoid. Suppose p satisfies the above relationship. Then since $b \neq 0$ we can set $v_0 = p_3/b$ and we obtain

$$p_1 \sin v_0 = p_2 \cos v_0 .$$

For any value of v_0 at least one of $\cos v_0$, $\sin v_0$ must be nonzero. Suppose $\cos v_0 \neq 0$. Then we can find $u_0 = p_1/\cos v_0$. Now $p_2 \cos v_0 = p_1 \sin v_0 = u_0 \cos v_0 \sin v_0$ and so, since $\cos v_0 \neq 0$, $p_2 = u_0 \sin v_0$. Hence we can write p in the form $(u_0 \cos v_0, u_0 \sin v_0, bv_0)$ and so it belongs to the helicoid.

A similar argument works when $\sin v_0 \neq 0$.

Hence the helicoid is the surface

$$M : x \sin(z/b) - y \cos(z/b) = 0.$$

2. Page 140, Exercise 2.

By Lemma II.1.8,

$$\|\mathbf{x}_u \times \mathbf{x}_v\|^2 = (\mathbf{x}_u \cdot \mathbf{x}_u)(\mathbf{x}_v \cdot \mathbf{x}_v) - (\mathbf{x}_u \cdot \mathbf{x}_v)^2$$
$$\doteq EG - F^2 .$$

Now **x** is regular if and only if $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$. But $\mathbf{x}_u \times \mathbf{x}_v$ is never zero if and only if $\|\mathbf{x}_u \times \mathbf{x}_v\|$ is never zero, which is true if and only if $EG - F^2$ is never zero.

3. Page 141, Exercise 6.

$$\mathbf{x}(u, v) = \beta(u) + v\mathbf{q}.$$

Hence $\mathbf{x}_u = \beta'$

and $\mathbf{x}_v = \mathbf{q}$:

$$\mathbf{x}_u \times \mathbf{u}_v = \beta' \times \mathbf{q},$$

and so **x** is regular if and only if $\beta' \times \mathbf{q}$ is never zero.

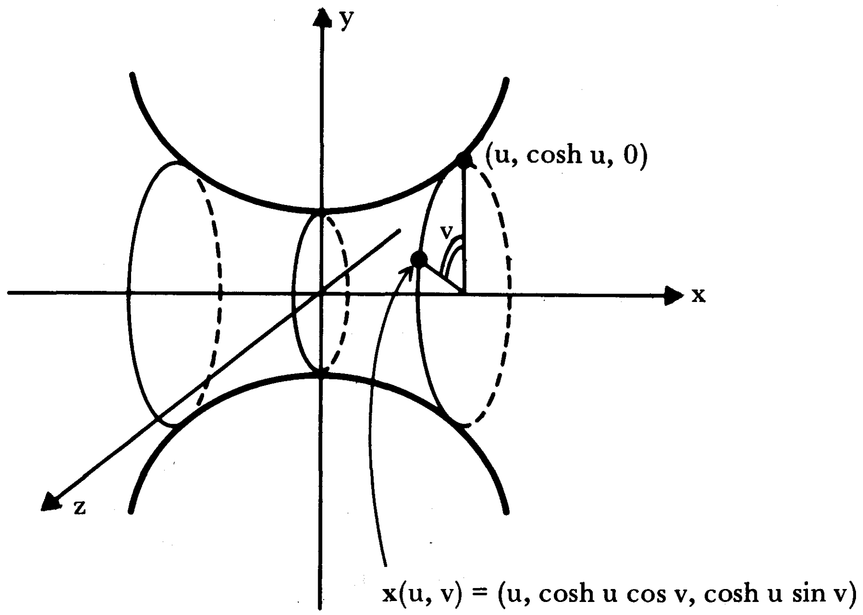
In Example 2.4 the curve β is in the xy plane and $\mathbf{q} = (0, 0, 1)$.

4. Page 140, Exercise 1.

(a) A parametrization of the profile curve is $u \longmapsto (u, \cosh u, 0)$.
A parametrization of the catenoid is

$$\mathbf{x} : (u, v) \longmapsto (u, \cosh u \cos v, \cosh u \sin v)$$

since rotation about the x axis keeps the x coordinate fixed.



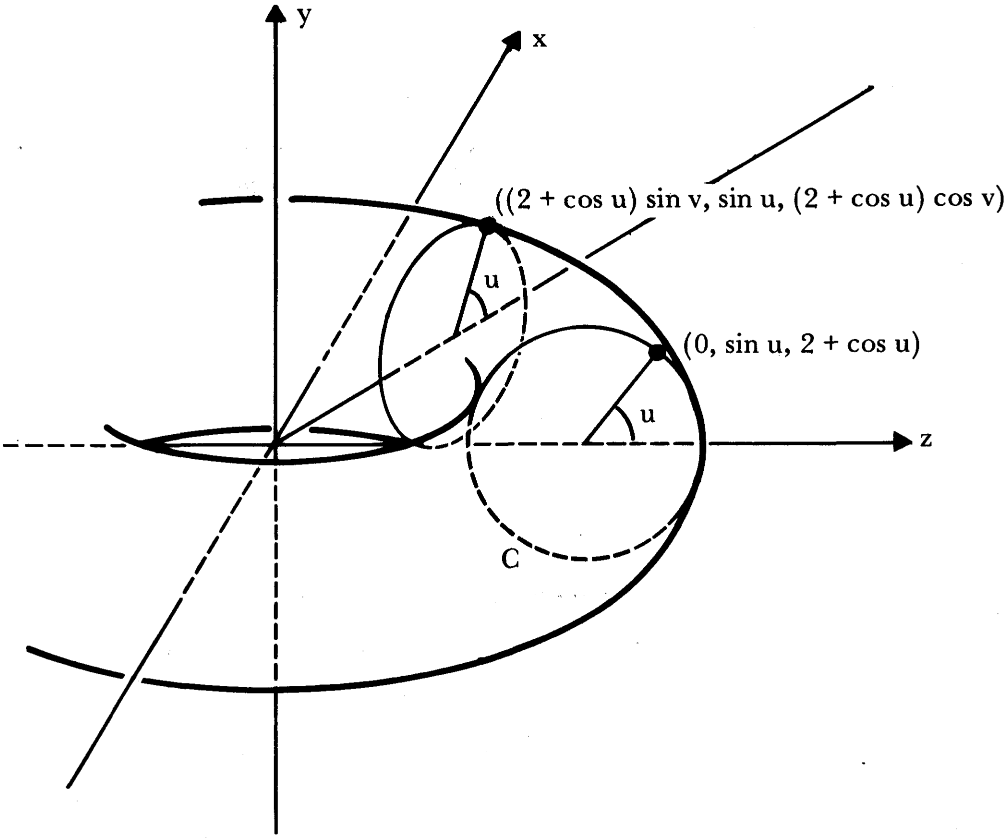
(b) C : $(z - 2)^2 + y^2 = 1$ is a circle in the yz plane of radius one and centre (0, 2). A possible parametrization is

$$u \longmapsto (0, \sin u, 2 + \cos u).$$

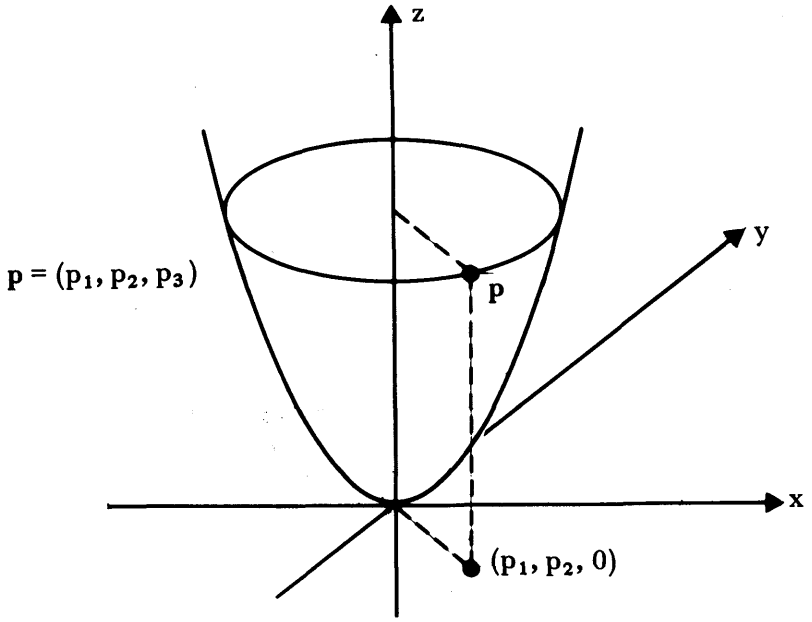
A parametrization of the torus is

$$x : (u, v) \longmapsto ((2 + \cos u) \sin v, \sin u, (2 + \cos u) \cos v),$$

since rotation about the y axis keeps the y coordinate fixed.



(c) We cannot follow the same procedure as in parts (a) and (b) since the profile curve cuts the axis of rotation and hence the standard ‘parametrization’ will not be regular. However we can describe the surface in terms of a Monge patch.



A point is on the paraboloid M if the square of its distance from the z axis is equal to its height above the xy plane. Thus $p = (p_1, p_2, p_3)$ belongs to M if and only if $p_1^2 + p_2^2 = p_3$. Hence $M : z = x^2 + y^2$ and we can parametrize it by mapping each point in the xy plane to the unique point on the surface above it. This gives the single patch

$$\mathbf{x} : (u, v) \longmapsto (u, v, u^2 + v^2).$$

5. Page 140, Exercise 4.

Look at the intersection of M with the plane $x = c$, for some constant c. This is the straight line $z = cy$ in the plane $x = c$. It passes through the points $(c, 0, 0)$ and $(c, 1, c)$. We can write it as

$$v \longmapsto (c, 0, 0) + v(0, 1, c).$$

Letting the point $(c, 0, 0)$ vary along the x axis we obtain the parametrization

$$(u, v) \longmapsto (u, 0, 0) + v(0, 1, u),$$

which is in ruled form.

Similarly looking at the intersection of M with planes of the form $y = c$ we find the parametrization

$$(u, v) \longmapsto (0, v, 0) + u(1, 0, v).$$

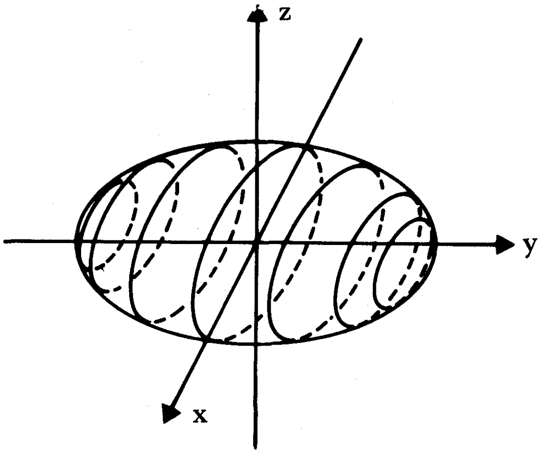
Alternatively, since $M : z = xy$, M is described by the Monge patch

$$\mathbf{x} : (u, v) \longmapsto (u, v, uv),$$

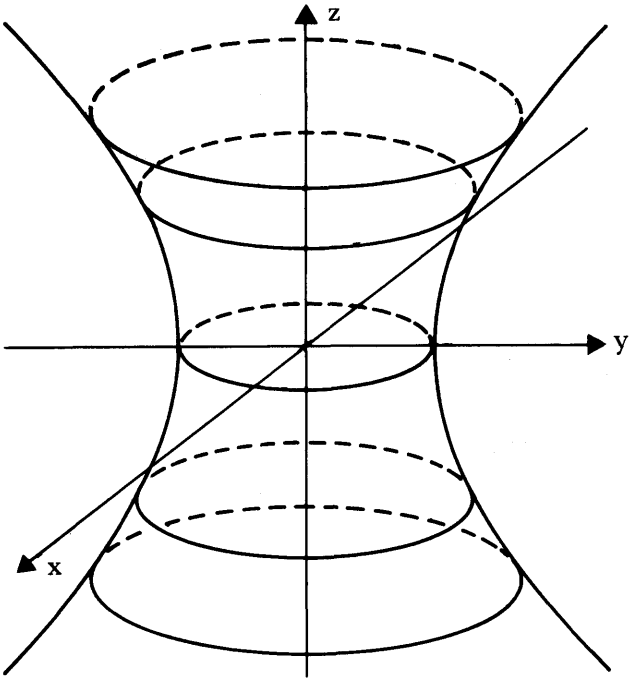
from which the above two ruled forms follow.

6. Page 142, Exercises 10 and 11.

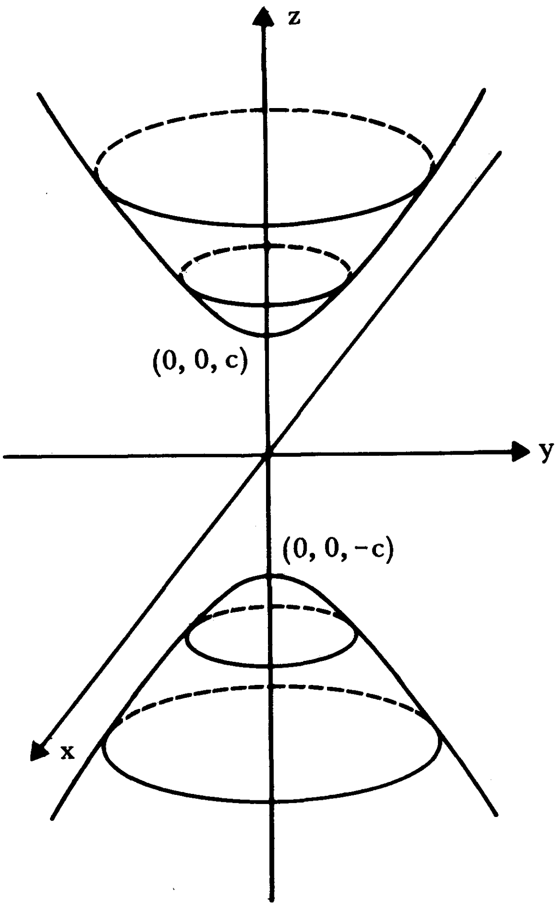
10(a). Ellipsoid



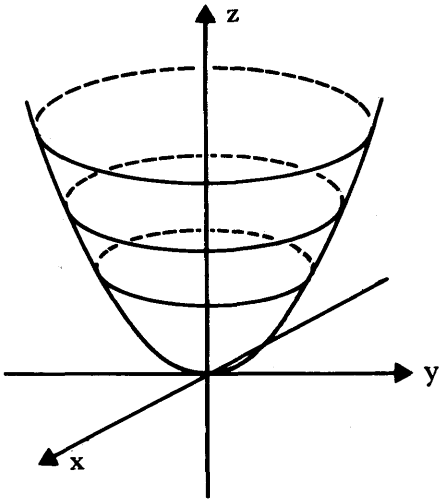
10(b). Elliptic hyperboloid



10(c). Elliptic hyperboloid (two sheets)



11. Elliptic paraboloid



7. Page 142, Exercise 12.

(a) The mapping $\mathbf{x} : E^2 \longrightarrow E^3$ does map into M since

$$4uv = \frac{(a(u+v))^2}{a^2} - \frac{(b(u-v))^2}{b^2}.$$

We need to show further that it maps onto M. For any point $\mathbf{p}$ belonging to M we need to solve the equations

$$p_1 = a(u_0 + v_0)$$

$$p_2 = b(u_0 - v_0)$$

$$p_3 = 4u_0 v_0.$$

We can solve the first two, obtaining

$$u_0 = \frac{p_1}{2a} + \frac{p_2}{2b}, \; v_0 = \frac{p_1}{2a} - \frac{p_2}{2b}$$

and since $\mathbf{p}$ belongs to M the third equation is also satisfied.

Next we need to check that $\mathbf{x}$ is one-to-one. Suppose

$$\mathbf{x}(u, v) = \mathbf{x}(u', v'):$$

then

$$u + v = u' + v'$$

$$u - v = u' - v'$$

and hence $u = u'$, $v = v'$. The mapping $\mathbf{x}$ is one-to-one.

Now we need to check that $\mathbf{x}$ is regular: we use the usual method.

$$\mathbf{x}_u \times \mathbf{x}_u = \begin{vmatrix} U_1 & U_2 & U_3 \\ a & b & 4v \\ a & -b & 4u \end{vmatrix}$$

$$= (4b(u+v), 4a(v-u), -2ab).$$

Now, for the definition of the hyperbolic paraboloid to be meaningful we need $a, b > 0$. Hence $-2ab \neq 0$ and so $\mathbf{x}_u \times \mathbf{x}_v$ is never zero. This in turn proves that $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent and hence $\mathbf{x}$ is regular.

Thus $\mathbf{x}$ is indeed a patch and to complete the proof we need only prove that $\mathbf{x}$ is proper. To prove that $\mathbf{x}$ is proper we have to show that the inverse function $\mathbf{x}^{-1} : M \longmapsto E^2$ is continuous. Since $\mathbf{x}^{-1}$ is given by the formula

$$\left(\frac{x}{2a} + \frac{y}{2b}, \frac{x}{2a} - \frac{y}{2b}\right)$$

continuity follows from standard results in analysis.

(b) We can write $\mathbf{x}$ in ruled form as follows

(i) $\mathbf{x}(u, v) = (au, bu, 0) + v(a, -b, 4u)$

and

(ii) $\mathbf{x}(u, v) = (av, -bv, 0) + u(a, b, 4v)$.

IV.3 DIFFERENTIABLE FUNCTIONS AND TANGENT VECTORS

Introduction

This section follows on from the previous section, IV.2, and follows the development of Chapter I, Sections 1 to 4. You will also need the chain rule dealt with in Section I.7.

The section begins with certain rather technical but intuitively obvious definitions and results. Mappings that you would expect to be differentiable turn out to be differentiable, and, given a coordinate patch, we can use it to give explicit descriptions of curves in a surface. Do not worry about the technicalities.

We then show that the partial velocities, for a given patch, do tell us something independent of the choice of patch. The tangents to any curve in the surface can be written as a linear combination of them. These tangents form the ***tangent planes*** to the surface. We use them to define a ***directional derivative*** for functions defined only on the surface, and not on the whole of E^3.

READ: Section IV.3 (pages 143 - 149). Do not spend much time on pages 143 - 145. Just try to understand the results.

Comments

(i) *Differentiation on M and on E^3* We have now met several different definitions of differentiation with domain or codomain either E^3 or some surface M. If we are dealing with a function $F : E^n \longrightarrow E^m$ between Euclidean spaces and we are using the definition of differentiability given in Section I.7 we say that F is Euclidean differentiable. If we are using either of the definitions of this section for functions with domain or codomain some surface M we indicate it by saying that $G : M \longrightarrow E^n$ is differentiable or $F : E^n \longrightarrow M$ is differentiable. We need to compare these definitions of differentiability with the Euclidean definition.

Directly from the definitions we have that $G : M \longrightarrow E^n$ or $F : E^n \longrightarrow M$ are differentiable if their respective coordinate expressions are Euclidean differentiable.

Suppose we have a function $F : E^n \longrightarrow M$. Then we can also consider it as a function $F : E^n \longrightarrow E^3$. If we know that F is differentiable as a function from E^n to M then we would hope that F is also Euclidean differentiable. It is. Since $F : E^n \longrightarrow M$ is differentiable it follows, by definition, that $\mathbf{x}^{-1} \circ F : E^n \longrightarrow E^2$ is Euclidean differentiable, for each patch **x**. But by the definition of a patch, each patch $\mathbf{x} : D \longrightarrow E^3$ is Euclidean differentiable. Consequently the composite mappings $\mathbf{x} \circ \mathbf{x}^{-1} \circ F : E^n \longrightarrow E^3$ are Euclidean differentiable. But each of these mappings is just the restriction of F to those points whose image under F is in **x**(D). Since the patches cover M we can choose enough "**x**"s to deal with F at every point. Hence $F : E^n \longrightarrow E^3$ is Euclidean differentiable.

Theorem 3.2 gives us the converse of this result. If $F : E^n \longrightarrow E^3$ is a Euclidean differentiable mapping whose image lies in M then it is also differentiable considered as a function $F : E^n \longrightarrow M$.

The corollaries are proved as follows.

Corollary 3.3 Since y is a patch it is a differentiable function $y : D \longrightarrow E^3$ whose image lies in M. Hence, by Theorem 3.2, it is also differentiable considered as a function $y : D \longrightarrow M$. By definition this just means that $x^{-1} \circ y$ is a Euclidean differentiable mapping between suitable subsets of E^2.

Corollary 3.4 Since $x^{-1}y$ is Euclidean differentiable

$$x^{-1}y(u, v) = (\bar{u}(u, v), \bar{v}(u, v)),$$

where the Euclidean coordinate functions $\bar{u}$, $\bar{v}$ are differentiable. Applying the mapping x to each side gives the result. The mapping $x^{-1}\ y = (\bar{u}, \bar{v})$ gives us a coordinate transformation.

Theorem 3.2 can also be used to give a converse of *Lemma 3.1.* Suppose a function $\alpha : I \longrightarrow M$ is such that

$$\alpha(t) = x(a_1(t), a_2(t))$$

for differentiable functions a_1 and a_2. Then the composite $x(a_1, a_2)$ is a Euclidean differentiable mapping into E^3, whose image lies in M. Hence by Theorem 3.2 its restriction $\alpha : I \longrightarrow M$ is differentiable; that is, α is a curve.

As a corollary to this we find that the parameter 'curves' are in fact curves in M since they can be written as

$$t \longmapsto x(t, v_0)$$

$$t \longmapsto x(u_0, t),$$

where the functions given by $t \longmapsto t$, $t \longmapsto v_0$, $t \longmapsto u_0$ are all differentiable.

When no confusion can arise we omit the word 'Euclidean' from the expression 'Euclidean differentiable'.

(ii) *Page 147: line 1* We have two mappings F and x with the following Euclidean coordinate functions:

$$F = (a_1, a_2) \quad \text{and} \quad x = (x_1, x_2, x_3).$$

The composite mapping $x \circ F$ is α. The composite (chain) rule gives

$$\alpha_* = x_*(F) \circ F_*.$$

In terms of Jacobian matrices this becomes

$$\begin{pmatrix} \alpha_1' \\ \alpha_2' \\ \alpha_3' \end{pmatrix}_t = \begin{pmatrix} \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_1}{\partial v} \\ \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_2}{\partial v} \\ \dfrac{\partial x_3}{\partial u} & \dfrac{\partial x_3}{\partial v} \end{pmatrix}_{F(t)} \begin{pmatrix} a_1' \\ a_2' \end{pmatrix}_t$$

Multiplying out we obtain

$$\begin{aligned} \alpha'(t) &= x_u(F(t))a_1'(t) + x_v(F(t))a_2'(t) \\ &= x_u(a_1(t), a_2(t))a_1'(t) + x_v(a_1(t), a_2(t))a_2'(t). \end{aligned}$$

(iii) *Page 147: vector fields* Since $x_u(u_0, v_0)$ and $x_v(u_0, v_0)$ are tangent vectors at $x(u_0, v_0)$ we can consider x_u and x_v as tangent vector fields on x(D). Since $x_u \times x_v$ is orthogonal to both x_u and x_v we can consider it as a nonvanishing normal vector field on x(D). It may be impossible to extend this to a nonvanishing normal vector field on the whole of M.

(iv) *Page 148: the proof of Lemma 3.8* It is possible to have implicitly defined subsets $M : g = c$ in E^3 that are surfaces but for which dg and hence the derivatives $\partial g/\partial x_i$ are zero. An example of such a subset is $M : x^2 = 0$. The only nontrivial derivative is $\partial g/\partial x = 2x$, which is identically zero on M. Hence the converse of Theorem IV.1.4 is false. In this lemma we need to add the restriction that dg is nonzero on M.

The form of the composite (chain) rule used is as follows. We have

$$\alpha : t \longmapsto (\alpha_1(t), \alpha_2(t), \alpha_3(t))$$

$$g : (p_1, p_2, p_3) \longmapsto g(p),$$

and $g\alpha : t \longmapsto c$ (constant).

Hence

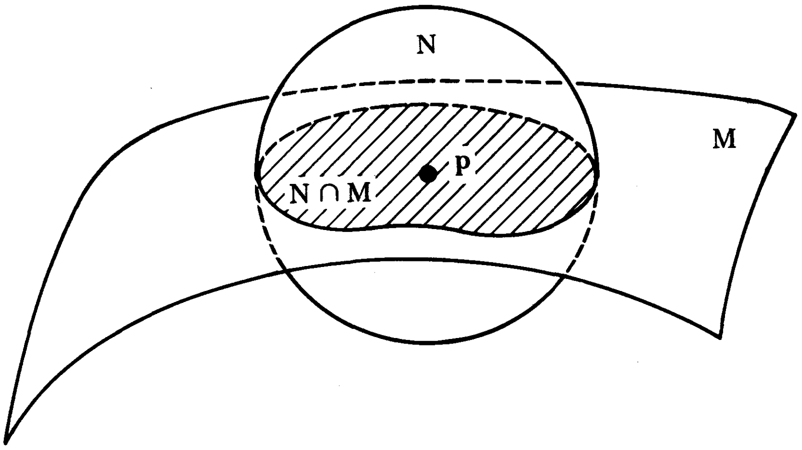
$$0 = (g\alpha)'(t) = \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}\right)_{\alpha(t)} \begin{pmatrix} \alpha_1' \\ \alpha_2' \\ \alpha_3' \end{pmatrix}_t$$

which gives the required result.

(v) *Page 149: Definition 3.10* We have defined the directional derivative, v[f], of a function f on M as the common value of $(d(f\alpha)/dt)(0)$ for all curves α in M with initial velocity v.

It is not at all obvious that this directional derivative is well defined. Given any vector v tangent to M we first of all have to assure ourselves that there is a curve α with initial velocity v and then we have to assure ourselves that the value of $(d(f\alpha)/dt)(0)$ is independent of which curve α we choose. The first point is easily settled. Since v is a tangent to M, by Definition 3.5 it is the velocity vector of some curve in M and by suitably shifting the parametrization of the curve we can ensure that it is the initial velocity of a curve α in M.

The second point is rather more complicated. To apply Theorem I.4.6 we need to extend f to a differentiable function F defined on some neighbourhood N of p in E^3.



If q is in the intersection of N and M then f(q) = F(q).

It is always possible to extend the function f in this way and then we can apply results on directional derivatives in E^3 to the function F. Of course, we need to be even more careful since it is possible to choose F in many distinct ways. At this stage we can use Theorem I.4.6. It tells us that

$$v[F] = \beta'(0)[F] = \frac{d(F\beta)}{dt}(0),$$

for any curve β with initial velocity v. Hence

$$v[F] = \frac{d(F\alpha)}{dt}(0)$$

for any curve α in M with initial velocity v. Now since F is an extension of f we find that

$$v[F] = \frac{d(f\alpha)}{dt}(0)$$

for any curve α in M with initial velocity v. Looking at the left hand expression we see that it does not mention any curve α and so is independent of that choice. Looking at the right hand expression we see that it does not mention any extension F and so is independent of that choice. Since the left hand expression equals the right hand expression and one is independent of the choice of α and the other is independent of the choice of F it follows that they are both independent of the choice of either α or F. Hence the expressions $(d(f\alpha)/dt)(0)$ do have a common value and we can define

$$v[f] = \frac{d(f\alpha)}{dt}(0).$$

This argument also suggests an alternative method of evaluating v[f]. We have seen that

$$v[f] = \frac{d(f\alpha)}{dt}(0) = \frac{d(F\alpha)}{dt}(0) = v[F]$$

for any suitable extension F. Hence if we can extend f, and in most cases this is automatic, we can evaluate v[f] as v[F] and use all the techniques developed in Section I.3.

It follows that this new directional derivative satisfies the usual linear and Leibnizian properties (Theorem I.3.3) when we restrict attention to functions on M and vectors tangent to M.

We shall need to find directional derivatives with respect to the partial velocities $x_u(u_0, v_0)$ and $x_v(u_0, v_0)$. Now $x_u(u_0, v_0)$ is the initial velocity of the curve $\alpha(t) = x(u_0 + t, v_0)$ and so

$$x_u(u_0, v_0)[f] = \frac{d(fx(u_0 + t, v_0))}{dt}(0) = \frac{\partial(fx)}{\partial u}(u_0, v_0).$$

Similarly

$$x_v(u_0, v_0)[f] = \frac{\partial(fx)}{\partial v}(u_0, v_0).$$

Letting the point (u_0, v_0) vary throughout the domain of the patch $\mathbf{x}$ we find that

$$\mathbf{x}_u[f] = \frac{\partial(f\mathbf{x})}{\partial u} \quad \text{and} \quad \mathbf{x}_v[f] = \frac{\partial(f\mathbf{x})}{\partial v}.$$

In the usual pointwise manner we can extend the definition of the directional derivatives to apply to any tangent vector field V on M. For any real differentiable function f on M we define a new differentiable function $V[f]$ as follows:

$$V[f](\mathbf{p}) = V(\mathbf{p})[f],$$

for any point $\mathbf{p}$ in M.

Summary

Notation

$\alpha(t) = \mathbf{x}(a_1(t), a_2(t))$ $\alpha = \mathbf{x}(a_1, a_2)$	Page 144, Lemma 3.1
$T_{\mathbf{p}}(M)$	Page 146, line 15
∇g	Page 148, Lemma 3.8
$\mathbf{v}[f]$	Page 149, Definition 3.10

Definitions

(i)	Coordinate expression $f(\mathbf{x})$ or $\mathbf{x}^{-1}(F)$	Page 143, line −9 and Page 143, line −5
(ii)	Differentiable functions on or to M	Page 143, line 8 and Page 143, line −3
(iii)	Curve in a surface	Page 144, line 3
(iv)	Coordinate expressions for a curve in a surface: $\alpha(t) = \mathbf{x}(a_1(t), a_2(t))$ or $\alpha = \mathbf{x}(a_1, a_2)$	Page 144, Lemma 3.1
(v)	Coordinate transformation, $\mathbf{x}^{-1}\mathbf{y} = (\bar{u}, \bar{v})$	Text, page 34, line 9
(vi)	Tangent to M at $\mathbf{p}$	Page 146, Definition 3.5
(vii)	Tangent plane $T_{\mathbf{p}}(M)$	Page 146, line 14
(viii)	Euclidean vector field on M	Page 147, Definition 3.7
(ix)	Tangent vector field on M	Page 147, line −10
(x)	Vector normal to M	Page 147, line −6
(xi)	Normal vector field on M	Page 147, line −4
(xii)	Gradient ∇g	Page 148, Lemma 3.8
(xiii)	Directional derivative ($\mathbf{v}[f]$) of a function on M	Page 149, Definition 3.10

Example

(i)	$\Sigma x_i U_i$ is a normal vector field on the sphere $\Sigma : g = \Sigma x_i^2 = r^2.$	Page 148, Example 3.9

Results

(i)	Any curve α in M whose image lies in $\mathbf{x}(D)$ can be expressed as $\alpha(t) = \mathbf{x}(a_1(t), a_2(t))$.	Page 144, Lemma 3.1
(ii)	If $F : E^n \longrightarrow E^3$ is differentiable and its image is in M then $F : E^n \longrightarrow M$ is differentiable.	Page 145, Theorem 3.2
(iii)	If $\mathbf{x}$, $\mathbf{y}$ are patches then $\mathbf{x}^{-1}\mathbf{y}$ and $\mathbf{y}^{-1}\mathbf{x}$ are differentiable.	Page 145, Corollary 3.3
(iv)	If $\mathbf{x}$, $\mathbf{y}$ are overlapping patches there are differentiable functions $(\bar{u}, \bar{v})$ such that $\mathbf{y}(u, v) = \mathbf{x}(\bar{u}(u, v), \bar{v}(u, v))$.	Page 145, Corollary 3.4
(v)	$\mathbf{v}$ is a tangent vector to M at $\mathbf{p} = \mathbf{x}(u_0, v_0)$ if and only if $\mathbf{v}$ is a linear combination of $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$.	Page 146, Lemma 3.6
(vi)	If α is a curve in M such that $\alpha = \mathbf{x}(a_1, a_2)$, then $\alpha' = \mathbf{x}_u(a_1, a_2)a_1' + \mathbf{x}_v(a_1, a_2)a_2'$.	Page 147, line 1
(vii)	If $M : g = c$ is a surface and dg is never zero on M then the gradient ∇g is a nonvanishing normal vector field.	Page 148, Lemma 3.8
(viii)	$\mathbf{x}_u \times \mathbf{x}_v$ is a normal vector field on $\mathbf{x}(D)$.	Text, page 35, line 3
(ix)	For any patch $\mathbf{x}$: $\mathbf{x}_u[f] = \frac{\partial}{\partial u}(f\mathbf{x})$ and $\mathbf{x}_v[f] = \frac{\partial}{\partial v}(f\mathbf{x})$.	Text, page 37, line 3

Techniques

(i)	Deciding whether a function to or from M is differentiable by finding its coordinate expressions.	Page 143, line −9, Page 143, line −5, and Page 144, Lemma 3.1
(ii)	Finding coordinate transformations between patches.	Page 145, Corollary 3.4
(iii)	Determining whether a vector is a tangent to M at $\mathbf{p} = \mathbf{x}(u_0, v_0)$ by checking whether it is a linear combination of $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$.	Page 146, Lemma 3.6
(iv)	Finding normal vector fields on M, $\mathbf{x}_u \times \mathbf{x}_v$ or ∇g, and using them to find vectors tangent to M.	Page 148, Lemma 3.8 and Text, page 35, line 3
(v)	Evaluating the directional derivatives of a function on a surface either from first principles, or by using the linear and Leibnizian properties, or by extending the function to some neighbourhood of the surface in E^3 and then applying standard techniques, or by using the formulas of Result (ix)	Page 149, Definition 3.10, Text, page 36, and Text, page 37

Exercises

Technique (i)

1. Page 149, Exercise 1.
2. Page 149, Exercise 2.

Technique (ii)

3. Page 150, Exercise 6.

Technique (iii)

4. Page 150, Exercise 8(a).

Technique (iv)

5. Page 150, Exercise 5.

Technique (v)

6. Calculate $\mathbf{x}_v[f]$ where $f(x, y, z) = z$ and $\mathbf{x}$ is the geographical patch in the sphere.

Solutions

1. Page 149, Exercise 1.

The geographical patch is

$$\mathbf{x}(u, v) = (r \cos v \cos u, r \cos v \sin u, r \sin v).$$

(a) If $f(p) = p_1^2 + p_2^2$ then the coordinate expression is given by the composite

$$(u, v) \xmapsto{\mathbf{x}} \mathbf{x}(u, v) \xmapsto{f} r^2 \cos^2 v \cos^2 u + r^2 \cos^2 v \sin^2 u = r^2 \cos^2 v.$$

(b) If $f(p) = (p_1 - p_2)^2 + p_3^2$ then the coordinate expression is

$$\begin{aligned}(u, v) \longmapsto \mathbf{x}(u, v) \longmapsto & r^2 \cos^2 v(\cos u - \sin u)^2 + r^2 \sin^2 v \\ &= r^2 \cos^2 v(1 - 2\cos u \sin u) + r^2 \sin^2 v \\ &= r^2 (1 - 2\cos^2 v \cos u \sin u) \\ &= r^2 (1 - \cos^2 v \sin 2u).\end{aligned}$$

2. Page 149, Exercise 2.

The parametrization of the torus is

$$\mathbf{x}(u, v) = ((R + r \cos u)\cos v, (R + r \cos u)\sin v, r \sin u).$$

(a) We are given the coordinate functions for the curve α. They are $a_1(t) = a_2(t) = t$.

Hence

$$\begin{aligned}\alpha(t) &= \mathbf{x}(a_1(t), a_2(t)) = \mathbf{x}(t, t) \\ &= ((R + r \cos t) \cos t, (R + r \cos t) \sin t, r \sin t).\end{aligned}$$

(b) If $\alpha(t) = \alpha(t')$ then $\sin t = \sin t'$. If as well $\sin t = \sin t' \neq 0$ then $R + r\cos t = R + r\cos t'$ and hence $\cos t = \cos t'$. If on the contrary $\sin t = \sin t' = 0$ then $\cos t$, $\cos t' = \pm 1$. Since $(R + r\cos t)\cos t = (R + r\cos t')\cos t'$ and $r < R$ it follows that either $\cos t = 1 = \cos t'$ or $\cos t = -1 = \cos t'$. In any case both $\sin t = \sin t'$ and $\cos t = \cos t'$ and so $t - t' = 2\pi n$ for some integer n.

Conversely, if $t - t' = 2\pi n$ then $\alpha(t) = \alpha(t')$ since the trigonometric functions are periodic with period 2π.

Hence $\alpha(t) = \alpha(t')$ if and only if $t - t' = 2\pi n$ for some integer n; that is, α is periodic with period 2π.

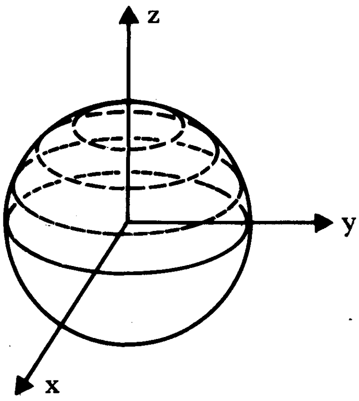
3. Page 150, Exercise 6.

(a)
$$x(u, v) = (u, v, (1-u^2-v^2)^{\frac{1}{2}})$$

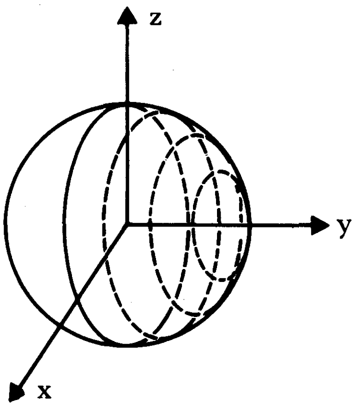
and

$$y(u, v) = (v, (1-u^2-v^2)^{\frac{1}{2}}, u).$$

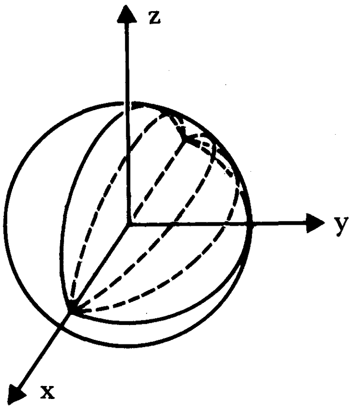
The first patch covers the northern hemisphere of the sphere, that is the hemisphere of points with strictly positive z coordinate. Similarly the second patch covers the hemisphere of points with strictly positive y coordinate. The intersection of these hemispheres is the region on the sphere for which both the z and y coordinates are strictly positive. These regions are shown in the following diagrams.



Region 1



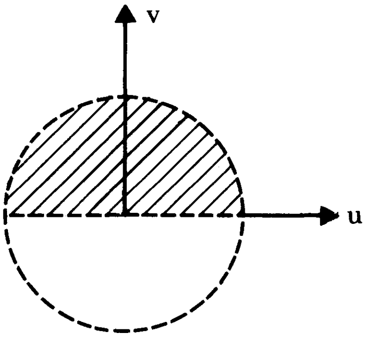
Region 2



Region 3

(b) $y^{-1}x$ is defined on points (u, v) belonging to D such that $x(u, v)$ belongs to region 2 or equivalently region 3. For this $x(u, v)$ must have positive y coordinate.

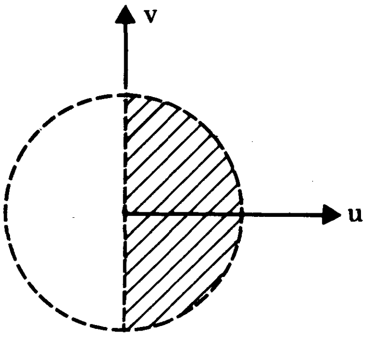
This happens when $v > 0$.



On this domain

$$y^{-1}x : (u, v) \xmapsto{x} (u, v, (1-u^2-v^2)^{\frac{1}{2}}) \xmapsto{y^{-1}} ((1-u^2-v^2)^{\frac{1}{2}}, u).$$

(c) A similar argument to that in (b) shows that $x^{-1}y$ is defined on points in D for which $y(u, v)$ has positive z coordinate. This happens if $u > 0$.



On this domain

$$x^{-1}y \; : \; (u, v) \longmapsto (v, (1-u^2-v^2)^{\frac{1}{2}}).$$

4. Page 150, Exercise 8(a).

By Result (vi), if $\alpha = x(a_1, a_2)$ is a curve in a surface then

$$\alpha' = x_u(a_1, a_2)a_1{}' + x_v(a_1, a_2)a_2{}'.$$

In this exercise $a_1(t) = \sqrt{2}t$, $a_2(t) = e^t$ and hence

$$\alpha'(t) = \sqrt{2}\, x_u(\sqrt{2}t, e^t) + e^t\, x_v(\sqrt{2}t, e^t).$$

5. Page 150, Exercise 5.

(a) M can be defined implicitly as

$$M : g(x, y, z) = z - f(x, y) = 0.$$

Hence by Lemma IV.3.8 we know that

$$\nabla g = \frac{\partial g}{\partial x} U_1 + \frac{\partial g}{\partial y} U_2 + \frac{\partial g}{\partial z} U_3 = -\frac{\partial f}{\partial x} U_1 - \frac{\partial f}{\partial y} U_2 + U_3$$

is a normal vector field on M.

Hence a tangent vector **v** at **p** is a tangent vector to M at **p** if and only if

$$\begin{aligned} 0 &= \nabla g(p) \cdot v \\ &= \left(-\frac{\partial f}{\partial x}(p_1, p_2), -\frac{\partial f}{\partial y}(p_1, p_2), 1\right) \cdot (v_1, v_2, v_3) \\ &= -v_1 \frac{\partial f}{\partial x}(p_1, p_2) - v_2 \frac{\partial f}{\partial y}(p_1, p_2) + v_3, \end{aligned}$$

which is the required condition.

(b) Since the vector $x_u(u_0, v_0) \times x_v(u_0, v_0)$ is nonzero and normal to M at $p = x(u_0, v_0)$, the vector v at p is tangent to M if and only if

$$v \cdot (x_u(u_0, v_0) \times x_v(u_0, v_0)) = 0.$$

6. By Result (ix) we know that

$$\mathbf{x}_v[f] = \frac{\partial}{\partial v}(f\mathbf{x}).$$

Now

$$(f\mathbf{x})(u, v) = r \sin v$$

and hence

$$\mathbf{x}_v[f] = r \cos v.$$

That is,

$$\mathbf{x}_v(u_0, v_0)[f] = r \cos v_0 .$$

IV.4 DIFFERENTIAL FORMS ON A SURFACE

Introduction

This section follows on from Section IV.3 and is closely related to Section I.6.

In Section I.6 we described differential forms on E^3 in terms of coordinates. In this section we shall describe differential forms on a surface in a coordinate independent way. The results that we shall obtain parallel those of Section I.6.

READ: Section IV.4 (pages 152–157) omitting the proof of Lemma 4.5.

Comments

(i) *Differential Forms* A 2-form η operates only on pairs of tangent vectors with the same point of application. If v_p, w_p are vectors tangent to M at p then the real number $\eta(v_p, w_p)$ is defined.

Condition (1), of Definition 4.1, says that η restricted to any one tangent space $T_p(M)$ is a bilinear form, and condition (2) says that it is anti-symmetric. If **u**, **v**, **w** are tangent vectors to M at p and a, b, are real numbers then condition (1) is equivalent to

$$\eta(au + bv, w) = a\eta(u, w) + b\eta(v, w)$$

and

$$\eta(u, av + bw) = a\eta(u, v) + b\eta(u, w).$$

As well as using the pointwise principle to define addition of forms we can use it to define the multiplication of a form by a real valued function f on M. If ϕ is a 1-form and η a 2-form on M and v, w are tangent vectors to M at p then we can define forms $f\phi$ and $f\eta$ as follows:

$$(f\phi)(v) = f(p)\,\phi(v)$$

and

$$(f\eta)(v, w) = f(p)\,\eta(v, w).$$

We can also use the pointwise principle to define the effect of forms on tangent vector fields. If V, W are tangent vector fields on M and ϕ is a 1-form and η a 2-form then the real valued functions, $\phi(V)$ and $\eta(V, W)$, on M are defined as follows:

$$(\phi(V))(p) = \phi(V(p))$$

and

$$(\eta(V, W))(p) = \eta(V(p), W(p)),$$

for each point p belonging to M.

Since $x_u(u_0, v_0)$ and $x_v(u_0, v_0)$ are a basis for the tangent vectors at the point $p = x(u_0, v_0)$, the 1-form ϕ and 2-form η are determined at p by the values of $\phi(x_u(u_0, v_0))$, $\phi(x_v(u_0, v_0))$ and $\eta(x_u(u_0, v_0), x_v(u_0, v_0))$ respectively.

We use the notation x_u and x_v to stand for the vector fields, on the image of the patch x, that take the values of $x_u(u_0, v_0)$ and $x_v(u_0, v_0)$ at the point $p = x(u_0, v_0)$. Hence on the image of the patch x the forms ϕ and η are determined by the real valued functions $\phi(x_u)$, $\phi(x_v)$ and $\eta(x_u, x_v)$.

We have defined 0-, 1-, and 2-forms on a surface but we do not yet know if they ever exist. In fact it is easy to construct examples. Since 0-forms are just differentiable functions on the surface they certainly exist. We can always at least choose constant functions.

Suppose we are dealing with a simple surface $M = x(D)$. We can define a 1-form ϕ as follows:

$$\phi : v \longmapsto v \cdot x_u(u_0, v_0),$$

where v is a vector tangent to M at $p = x(u_0, v_0)$.

Similarly we can define a 2-form η as follows:

$$\eta : v, w \longmapsto (v \times w) \cdot (x_u(u_0, v_0) \times x_v(u_0, v_0)),$$

where v, w are vectors tangent to M at $p = x(u_0, v_0)$. It is not difficult to see that ϕ and η, defined above, are non-trivial forms on $M = x(D)$.

(ii) *The wedge product* When we form the wedge product of two differentials we omit the wedge symbol. For instance we write $du_1 du_2$ rather than $du_1 \wedge du_2$.

Supplementary Comments

(i) *Page 153: after Definition 4.3* We need to check that $\phi \wedge \psi$ defined in this way really is a 2-form (an antisymmetric bilinear form on each tangent space). This is straightforward. For instance condition (2) is verified as follows. For any tangent vectors v, w at p,

$$\begin{aligned}(\phi \wedge \psi)(v, w) &= \phi(v)\,\psi(w) - \phi(w)\,\psi(v), && \text{by Definition 4.3,}\\ &= -\,(\phi(w)\,\psi(v) - \phi(v)\,\psi(w))\\ &= -\,((\phi \wedge \psi)(w, v)), && \text{by Definition 4.3.}\end{aligned}$$

(ii) *Page 154: line 2* For any pair of tangent vectors v, w at p,

$$\begin{aligned}(\phi \wedge \psi)\,(v, w) &= \phi(v)\,\psi(w) - \phi(w)\,\psi(v), && \text{by Definition 4.3,}\\ &= -\,(\psi(v)\,\phi(w) - \psi(w)\,\phi(v))\\ &= -\,((\psi \wedge \phi)(v, w)), && \text{by Definition 4.3,}\\ &= (-(\psi \wedge \phi))(v, w).\end{aligned}$$

Since this is true for all pairs v, w it follows that

$$\phi \wedge \psi = -\,\psi \wedge \phi.$$

Summary

Definitions

(i)	0-form on M : f,g . . .	Page 152, line 14 of the section
(ii)	1-form on M : ϕ, ψ . . .	Page 152, line 15 of the section
(iii)	2-form on M : η, ξ . . .	Page 152, Definition 4.1
(iv)	Pointwise addition of forms on M	Page 152, line −2
(v)	Pointwise multiplication of forms and functions on M	Text, page 43
(vi)	Pointwise effect of forms on tangent vector fields on M	Text, page 43
(vii)	The wedge product of 1-forms on M	Page 153, Definition 4.3
(viii)	The exterior derivative of a 0-form (function) on M	Page 154, line 4
(ix)	The exterior derivative of a 1-form on M	Page 154, Definition 4.4

Results

(i)	For any 2-form η on M and any vector v tangent to M then $\eta(v, v) = 0$.	Page 153, line 6
(ii)	If v, w are (linearly independent) vectors tangent to M at p and η is a 2-form on M then $\eta(av + bw, cv + dw) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \eta(v, w)$.	Page 153, Lemma 4.2
(iii)	If ϕ, ψ are 1-forms on M then $\phi \wedge \psi = - \psi \wedge \phi$.	Page 154, line 2
(iv)	The definition of the exterior derivative of a 1-form on M is independent of the choice of patches.	Page 154, Lemma 4.5
(v)	The exterior derivative of a wedge product satisfies the linear and Leibnizian properties.	Page 155, line 20
(vi)	If f is a differentiable function on M then $d(df) = 0$.	Page 155, Lemma 4.6
(vii)	1-forms that agree on $\mathbf{x}_u$ and $\mathbf{x}_v$ are equal on the patch $\mathbf{x}(D)$.	Page 156, line 10
(viii)	2-forms that agree on the pair $(\mathbf{x}_u, \mathbf{x}_v)$ are equal on the patch $\mathbf{x}(D)$.	Page 156, line 12

Technique

(i) The evaluation of linear combinations, wedge products and exterior derivatives of forms on a surface, using the above definitions and results.

Exercise

1. Page 156, Example 4.7

There are three ways we can extend the exterior calculus to deal with E^2.

(a) We can treat it formally as in Section I.6.

(b) We can describe 1- and 2-forms as operating on tangent vectors and pairs of tangent vectors in E^2. The wedge product is defined by analogy with Definition IV.4.3 and the exterior derivative by analogy with Definition IV.4.4 (replacing $\mathbf{x}_u$ and $\mathbf{x}_v$ by the natural frame field U_1 and U_2).

(c) We can consider E^2 as a surface in E^3 given by the single patch $\mathbf{x} : E^2 \longrightarrow E^3$

$$(u, v) \longmapsto (u, v, 0).$$

Check the results of Example IV.4.7 in each of these three cases. (In case (c) let u_1, u_2 be the restrictions of the coordinate functions x, y to $\mathbf{x}(E^2)$.)

Solution

1. (a)

(1) A 1-form is an expression $f_1 du_1 + f_2 du_2$, where f_1 and f_2 are (differentiable) functions.

(2) A 2-form is an expression $g du_1 du_2$.

(3) The wedge product is defined using linearity and the alternation rule so

$$\begin{aligned}\phi \wedge \psi &= (f_1 du_1 + f_2 du_2) \wedge (g_1 du_1 + g_2 du_2)\\ &= f_1 g_1 du_1 du_1 + f_1 g_2 du_1 du_2 + f_2 g_1 du_2 du_1 + f_2 g_2 du_2 du_2\\ &= (f_1 g_2 - f_2 g_1)\, du_1 du_2.\end{aligned}$$

(4) By analogy with Corollary I.5.5 we *define*

$$df = \frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2.$$

(5) We define the exterior derivative of a 1- form by finding the exterior derivative of its coordinate functions.

That is

$$\begin{aligned}d\phi &= df_1 \wedge du_1 + df_2 \wedge du_2\\ &= \left(\frac{\partial f_1}{\partial u_1} du_1 + \frac{\partial f_1}{\partial u_2} du_2\right) \wedge du_1 + \left(\frac{\partial f_2}{\partial u_1} du_1 + \frac{\partial f_2}{\partial u_2} du_2\right) \wedge du_2\\ &= \left(\frac{\partial f_2}{\partial u_1} - \frac{\partial f_1}{\partial u_2}\right) du_1 du_2.\end{aligned}$$

(b)

(1) A 1-form is a mapping on tangent vectors (and hence vector fields) that is linear on each tangent vector space. Hence two 1-forms ϕ and ψ are equal if they agree on the natural frame fields U_1 and U_2.

By analogy with Definition I.5.2 we define the differential in terms of the directional derivative and hence

$$du_i(U_j) = U_j[u_i] = \delta_{ij} \quad \text{for } i, j \in \{1, 2\}.$$

Let $\psi = \phi(U_1)du_1 + \phi(U_2)du_2$. Then

$$\psi(U_1) = \phi(U_1)du_1(U_1) + \phi(U_2)du_2(U_1) = \phi(U_1)$$

and

$$\psi(U_2) = \phi(U_1)du_1(U_2) + \phi(U_2)du_2(U_2) = \phi(U_2)$$

and so

$$\phi = \psi = \phi(U_1)du_1 + \phi(U_2)du_2.$$

(2) Two 2-forms are equal if they agree on the pair of vector fields (U_1, U_2), by analogy with Lemma IV.4.2.

By analogy with Definition IV.4.3 we have

$$du_1 du_2(U_1, U_2) = du_1(U_1)du_2(U_2) - du_1(U_2)du_2(U_1) = 1.$$

Let $\xi = \eta(U_1, U_2)du_1 du_2$. Then

$$\xi(U_1, U_2) = \eta(U_1, U_2)du_1 du_2(U_1, U_2) = \eta(U_1, U_2)$$

and so

$$\eta = \xi = \eta(U_1, U_2)\, du_1 du_2.$$

(3) Since the wedge product satisfies the alternation rule this follows as in part (a) (3).

(4) By analogy with Definition I.5.2,

$$df(U_i) = U_i[f] \quad \text{for } i = 1, 2.$$

But the directional derivative $U_i\,[f]$ is just $\partial f/\partial u_i$.

By result (b) (1),

$$\begin{aligned} df &= df(U_1)du_1 + df(U_2)du_2 \\ &= \frac{\partial f}{\partial u_1}du_1 + \frac{\partial f}{\partial u_2}du_2. \end{aligned}$$

(5) By analogy with Definition IV.4.4 we define

$$\begin{aligned} d\phi(U_1, U_2) &= \frac{\partial}{\partial u_1}(\phi(U_2)) - \frac{\partial}{\partial u_2}(\phi(U_1)) \\ &= \frac{\partial f_2}{\partial u_1} - \frac{\partial f_1}{\partial u_2}, \qquad \text{by result(b) (1).} \end{aligned}$$

Hence

$$d\phi = \left(\frac{\partial f_2}{\partial u_1} - \frac{\partial f_1}{\partial u_2}\right)du_1\,du_2, \qquad \text{by result (b) (2).}$$

(c) We consider E^2 as the surface in E^3 given by the single patch

$$\begin{aligned} \mathbf{x} : E^2 &\longrightarrow E^3 \\ (u, v) &\longmapsto (u, v, 0). \end{aligned}$$

Then $\mathbf{x}_u = U_1$, restricted to $\mathbf{x}(E^2)$, and $\mathbf{x}_v = U_2$, again restricted to $\mathbf{x}(E^2)$. We can also consider the Euclidean coordinate functions x and y restricted to $\mathbf{x}(E^2)$. We call these u_1 and u_2. By analogy with Definition I.5.2 and the definition of the directional derivative, Definition IV.3.10, we find from first principles that

$$du_i(U_j) = U_j[u_i] = \delta_{ij} \quad \text{for } i, j \in \{1,2\}.$$

(1) Since two 1-forms on a surface are equal if they agree on $\mathbf{x}_u$ and $\mathbf{x}_v$ we find, on applying both sides of equation (1) to U_1 and U_2, that

$$\phi = \phi(U_1)du_1 + \phi(U_2)du_2,$$

for any 1-form ϕ on $\mathbf{x}(D)$.

(2) Using the definition of the wedge product, Definition IV.4.3, and the result that two 2-forms are equal if they agree on the pair (U_1, U_2) we find as in (b) (2) that

$$\eta = \eta(U_1, U_2)du_1\,du_2$$

(3) This follows by the alternation property.

(4) We can evaluate the differential df on each of the partial velocities $U_1(u_0, v_0, 0)$ and $U_2(u_0, v_0, 0)$ at $(u_0, v_0, 0) = \mathbf{x}(u_0, v_0)$. We find that

$$\begin{aligned} df(U_1(u_0, v_0, 0)) &= U_1(u_0, v_0, 0)[f] \\ &= \frac{d}{dt}(f(u_0 + t, v_0, 0))\Big|_{t=0}. \end{aligned}$$

We shall write this as $\frac{\partial f}{\partial u_1}(u_0, v_0, 0)$ and $df(U_2(u_0, v_0, 0))$ as $\frac{\partial f}{\partial u_2}(u_0, v_0, 0)$.

From (c) (1)

$$df = df(U_1)du_1 + df(U_2)du_2$$

and hence

$$df = \frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2 .$$

(5) Combining the arguments of parts (b) (5) and (c) (4) we find that for any 1-form ϕ on $x(E^2)$

$$d\phi = \left(\frac{\partial f_2}{\partial u_1} - \frac{\partial f_1}{\partial u_2}\right) du_1 du_2 ,$$

where $f_1 = \phi(U_1)$ and $f_2 = \phi(U_2)$.

IV.5 MAPPINGS OF SURFACES

Introduction

This section follows on from Section IV.4.

Having defined surfaces and their differential calculus we next look at mappings between these structures. The mappings we choose are those that compose with the patches to give differentiable mappings. Given such a mapping F from M to N it is possible to take a tangent vector to M and map it to a tangent vector to N. By contrast a differential form on N can be mapped to a differential form on M. This allows us to relate the differential calculus on one surface with that on another in such a way that the various structures that we have defined are preserved. That is, the wedge product of two forms is mapped to the wedge product of the images of the forms, and the exterior derivative of a form is mapped to the exterior derivative of the image of the form.

We introduce certain mappings that became of great interest during the 18th and 19th centuries when mathematicians were interested in the problem of trying to make plane maps of our spherical globe.

READ: Section IV.5 (pages 158 - 164).

(i) *The derivative map* As usual we need to check that F_* is well defined. That is, given any tangent vector v to M and any curve α in M with initial velocity v we need to check that the initial velocity of the curve $F(\alpha)$ in N depends only on v and not on the choice of α.

We prove this in the same way that we proved that the directional derivative (Definition IV.3.10) was well defined. Suppose we extend F to a Euclidean differentiable function G defined on some neighbourhood in E^3 of the point in question. If α is a curve in M and $\beta = F(\alpha) = G(\alpha)$ then, by Theorem I.7.8, $\beta' = G_*(\alpha')$. So the initial velocity $\beta'(0) = G_*(\alpha'(0)) = G_*(v)$, where v is the initial velocity of α. Now $G_*(v)$ depends only on G and v and not on the choice of α while $\beta = F(\alpha)$ depends only on F and α and not on the extension G. Hence both sides of the equation are independent of the choice of curve α or extension G. Hence if we define $F_*(v) = \beta'(0) = (F\alpha)'(0)$ then it is well defined.

Given any vector v tangent to M then by definition we can find a curve α of which it is the initial velocity and so we do have a mapping F_* taking $T_p(M)$ to $T_{F(p)}(N)$. Since we have seen that F_* can be defined using the derivative map G_* of some extension and we saw in Section I.7 that such derivatives were linear it follows that the restriction F_* must also be a linear transformation. It also follows that if F_1 is a mapping from M_1 to M_2 and F_2 is a mapping from M_2 to M_3 then $(F_2 \circ F_1)_* = F_{2*} \circ F_{1*}$.

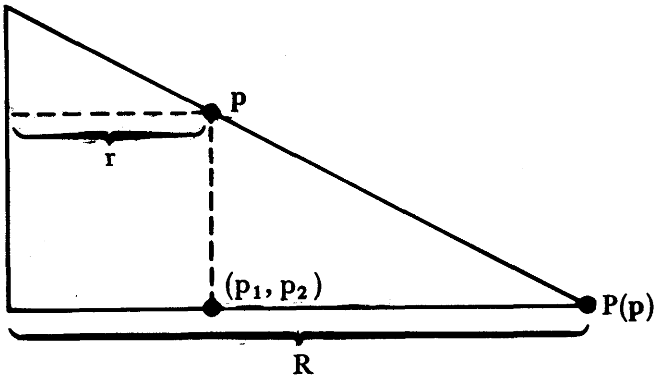
(ii) *F_* and F^** The positions of the stars on the symbols F_* and F^* are conventional. F is a map from M to N. Since tangent vectors are mapped in the same direction as F, from $T_p(M)$ to $T_{F(p)}(N)$, we use a lower star, F_*. Since differential forms are mapped in the opposite direction to F, that is forms on N are taken to forms on M, we use an upper star, F^*.

Since F_* is a linear transformation it is compatible with the vector space structures on the corresponding tangent spaces. Theorem 5.7 tells us that F^* is compatible with the operations of addition, wedge product and exterior derivative on the corresponding spaces of forms.

Supplementary Comments

(i) *Page 159: Example 5.2(1), line -8* The u-parameter measures the same angle for both the sphere and the cylinder. For the sphere the v-parameter measures the angle of elevation while for the cylinder it measures the height. Since F maps points radially onwards it preserves their height. A point on the sphere with v-parameter v_0 has height $\sin v_0$ and so is mapped by F to a point on the cylinder with v-parameter $\sin v_0$. The formula follows.

(ii) *Page 160: line 8*



By similar triangles $\frac{1}{R}P(p) = \frac{1}{r}(p_1, p_2)$.

Summary

Notation

F_*	Page 160, Definition 5.3
$F_{*p} : T_p(M) \longrightarrow T_{F(p)}(N)$	Page 161, line 12
F^*	Page 163, Definition 5.6

Definitions

(i)	Differentiable mapping between surfaces	Page 159, Definition 5.1
(ii)	Derivative map F_*	Page 160, Definition 5.3
(iii)	Regular	Page 161, line 11
(iv)	Diffeomorphism	Page 161, line -8
(v)	Induced mapping of forms F^*	Page 163, Definition 5.6

Examples

(i)	Horizontal projection of the sphere, minus poles, into the cylinder	Page 159, Example 5.2(1)
(ii)	Stereographic projection of the sphere, minus the north pole, onto the plane	Page 160, Example 5.2(2)

Results

(i)	F_* restricts to a linear transformation from $T_p(M)$ to $T_{F(p)}(N)$.	Page 160, line - 7
(ii)	The derivative map of a composite is the composite of the derivative maps; $(G \circ F)_* = G_* \circ F_*$.	Page 160, line - 2
(iii)	If x is a parametrization and $y = F(x)$ then $F_*(x_u) = y_u$ and $F_*(x_v) = y_v$.	Page 161, line 7
(iv)	If F is a one-to-one, onto and regular mapping then it is a diffeomorphism.	Page 162, line 1
(v)	For any mapping F the mapping F^*, between forms, is compatible with the operations of addition, wedge product and exterior derivative.	Page 163, Theorem 5.7

Techniques

(i)	Checking that a function is a mapping between surfaces by examining coordinate expressions.	Page 159, Definition 5.1
(ii)	Description of the derivative map in terms of partial velocities.	Page 161, line 7
(iii)	Showing that F is a diffeomorphism by checking that it is one-to-one, onto and regular.	Page 162, line 1
(iv)	Determination of the effect of F^* on forms.	Page 163, Definition 5.6, and Page 163, Theorem 5.7

Exercises

Technique (i)

1. Describe the mapping of Exercise 2(a), page 164, in terms of the geographical parametrization and hence describe its effect on meridians and parallels. (Restrict the domain of the parametrization to the region $-\frac{\pi}{2} < v < \frac{\pi}{2}$.)

Technique (ii)

2. Describe the derivative of the mapping in Exercise 1 in terms of the partial velocities associated with the usual geographical parametrization.

Technique (iii)

3. Check that the mapping of surfaces $P : \Sigma \longrightarrow E^2$ given in Example 5.5, page 162, is a diffeomorphism. (Use a Monge patch to prove that P is regular at the south pole.)

Technique (iv)

4. Page 165, Exercise 5. Find $A^*\gamma$, where γ is given by

$$\gamma(v_p, w_p) = p_3(v_1 w_2 - v_2 w_1).$$

5. Complete the proof of Theorem IV.5.7

Solutions

1. Page 164, Exercise 2(a).

$$F(p) = -p.$$

The geographical parametrization is

$$x(u, v) = r(\cos v \cos u, \cos v \sin u, \sin v).$$

Hence $F(x(u, v)) = r(-\cos v \cos u, -\cos v \sin u, -\sin v)$.

We want to describe the righthand side of this equation as $x(u', v')$. Now $\sin(-v) = -\sin v$, $\cos(-v) = \cos v$ and $\cos(u + \pi) = -\cos u$, $\sin(u + \pi) = -\sin u$. Hence $F(x(u, v)) = x(u + \pi, -v)$.

Hence the parallel given by the u-parameter curve $v = v_0$ is mapped to the parallel given by $v = -v_0$. A parallel in the northern hemisphere is mapped to the corresponding parallel in the southern hemisphere and vice versa.

The meridian given by the v-parameter curve $u = u_0$ is mapped to the v-parameter curve $u = u_0 + \pi$. That is, the meridians are rotated by 180°.

2. We must find the images of the two partial velocities at some point p and express these images in terms of the partial velocities at $F(p) = -p$. The geographical parametrization is

$$x(u, v) = r(\cos v \cos u, \cos v \sin u, \sin v).$$

The composite mapping $y = Fx = -x$ and hence

$$y(u, v) = -r(\cos v \cos u, \cos v \sin u, \sin v) = x(u + \pi, -v).$$

Suppose p is the point $x(u_0, v_0)$, then according to the result on page 161, line 7, the derivative $F_* : T_p(\Sigma) \longrightarrow T_{F(p)}(\Sigma)$ is given by

$$F_*(x_u(u_0, v_0)) = y_u(u_0, v_0)$$

and

$$F_*(x_v(u_0, v_0)) = y_v(u_0, v_0).$$

We want to express $y_u(u_0, v_0)$ and $y_v(u_0, v_0)$ in terms of the partial velocities of the geographical parametrization at their point of action $y(u_0, v_0) = x(u_0 + \pi, -v_0)$. That is, we want $y_u(u_0, v_0)$, $y_v(u_0, v_0)$ in terms of $x_u(u_0 + \pi, -v_0)$ and $x_v(u_0 + \pi, -v_0)$.

Now

$$\begin{aligned} y_u(u_0, v_0) &= (r \cos v_0 \sin u_0, -r \cos v_0 \cos u_0, 0) \\ &= (-r \cos(-v_0) \sin(u_0 + \pi), r \cos(-v_0) \cos(u_0 + \pi), 0) \\ &= x_u(u_0 + \pi, -v_0). \end{aligned}$$

Hence

$$F_*(x_u(u_0, v_0)) = x_u(u_0 + \pi, -v_0)$$

and a similar calculation shows that

$$F_*(x_v(u_0, v_0)) = -x_v(u_0 + \pi, -v_0).$$

Draw a picture for yourself: it may help you to understand the difference in sign of the two cases.

3. By Theorem IV.5.4 we need to check that P_* is one-to-one. For points other than the south pole this follows if we can show that y_u, y_v, where y is given on page 162, are linearly independent. Now

$$y_u = \left(\frac{-2\cos v\sin u}{(1-\sin v)}, \frac{2\cos v\cos u}{(1-\sin v)}\right)$$

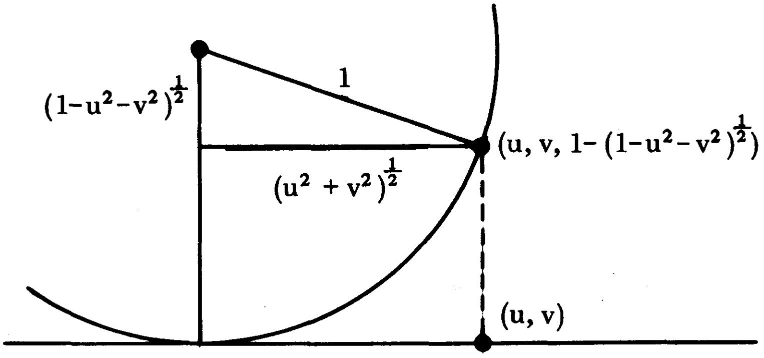
$$= \frac{2\cos v}{(1-\sin v)}(-\sin u, \cos u),$$

and

$$y_v = \left(\frac{2\cos u}{(1-\sin v)}, \frac{2\sin u}{(1-\sin v)}\right) = \frac{2}{(1-\sin v)}(\cos u, \sin u).$$

Since $v \in (-\frac{\pi}{2}, \frac{\pi}{2})$ it follows that $\cos v \neq 0$ and $\sin v \neq 1$. Hence y_u and y_v are well defined and nonzero. since the vectors $(\cos u, \sin u)$ and $(-\sin u, \cos u)$ are orthogonal y_u and y_v are linearly independent.

At the origin we use the following Monge patch:


$(1-u^2-v^2)^{\frac{1}{2}}$
1
$(u, v, 1-(1-u^2-v^2)^{\frac{1}{2}})$
$(u^2+v^2)^{\frac{1}{2}}$
(u, v)

$$x(u, v) = (u, v, 1-(1-u^2-v^2)^{\frac{1}{2}}).$$

By the formula on page 160,

$$Px(u, v) = \left(\frac{2u}{1+(1-u^2-v^2)^{\frac{1}{2}}}, \frac{2v}{1+(1-u^2-v^2)^{\frac{1}{2}}}\right) = y(u, v).$$

Differentiation yields

$$y_u(0, 0) = (1, 0)$$

and

$$y_v(0, 0) = (0, 1),$$

which are linearly independent. P is regular at the south pole and hence $P : \Sigma_0 \longrightarrow E^2$ is a diffeomorphism.

4. Page 165, Exercise 5.

We have already proved, using the geographical patch, that

$$A_*(x_u(u_0, v_0)) = x_u(\pi + u_0, -v_0)$$

and

$$A_*(x_v(u_0, v_0)) = -x_v(\pi + u_0, -v_0).$$

If $v_p = x_u(u_0, v_0) = (-r\cos v_0 \sin u_0, r\cos v_0\cos u_0, 0)$ then

$$x_u(\pi + u_0, -v_0) = (r\cos v_0\sin u_0, -r\cos v_0\cos u_0, 0) = (-v)_{-p}.$$

Similarly if $v_p = x_v(u_0, v_0)$ then $-x_v(\pi + u_0, -v_0) = (-v)_{-p}$. Since $x_u(u_0, v_0)$ and $x_v(u_0, v_0)$ form a basis for the tangent vectors at $p = x(u_0, v_0)$ it follows that

$$A_*(v_p) = (-v)_{-p}.$$

This holds for each point p of the geographical patch. Since we can find similar patches to cover the remainder of Σ it holds for all points p in Σ. Hence A_* is one-to-one and A is a diffeomorphism.

We can prove that $A_*(v_p) = (-v)_{-p}$ more simply by extending the mapping A to all of E^3. Since $A(p_1, p_2, p_3) = (-p_1, -p_2, -p_3)$ it follows that A_* has Jacobian matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Hence $A_*(v_p) = (-v)_{-p}$, for all tangent vectors to E^3, and so restricting A to Σ we still have

$$A_*(v_p) = (-v)_{-p}.$$

Now if a 2-form is defined on Σ by

$$\gamma(v_p, w_p) = p_3(v_1 w_2 - v_2 w_1)$$

then $A^*\gamma$ is given by

$$\begin{aligned} A^*\gamma(v_p, w_p) &= \gamma(A_*(v_p), A_*(w_p)) \\ &= \gamma((-v)_{-p}, (-w)_{-p}) \\ &= (-p_3)((-v_1)(-w_2) - (-v_2)(-w_1)) \\ &= -p_3(v_1 w_2 - v_2 w_1) \\ &= -\gamma(v_p, w_p). \end{aligned}$$

Hence $A^*\gamma = -\gamma$.

5. (1) The proofs are routine. For instance if ξ and η are 1-forms then for any vector v tangent to M

$$\begin{aligned} (F^*(\xi + \eta))(v) &= (\xi + \eta)(F_*v), && \text{by the definition of } F^*, \\ &= \xi(F_*v) + \eta(F_*v), && \text{by the definition of the addition of forms,} \\ &= (F^*\xi)(v) + (F^*\eta)(v) \\ &= (F^*\xi + F^*\eta)(v), \end{aligned}$$

and hence $F^*(\xi + \eta) = F^*\xi + F^*\eta$.

(2) This result also follows automatically. For instance if ξ and η are 1-forms then for any pair of vectors v, w tangent to M at the same point

$$\begin{aligned} (F^*(\xi \wedge \eta))(v, w) &= (\xi \wedge \eta)(F_*v, F_*w) \\ &= \xi(F_*v)\eta(F_*w) - \xi(F_*w)\eta(F_*v) \\ &= F^*\xi(v)\, F^*\eta(w) - F^*\xi(w) F^*\eta(v) \\ &= (F^*\xi \wedge F^*\eta)(v, w) \end{aligned}$$

and hence $F^*(\xi \wedge \eta) = F^*\xi \wedge F^*\eta$.

If f is a 0-form and ϕ is a 1-form then for any tangent vector v at p

$$\begin{aligned}(F^*(f\phi))(v) &= (f\phi)(F_*(v))\\ &= f(F(p))\,\phi(F_*(v))\\ &= (F^*f)(p)(F^*\phi)(v)\\ &= ((F^*f)(F^*\phi))(v)\end{aligned}$$

and hence $F^*(f\phi) = (F^*f)\,(F^*\phi)$.

(3) Since we have already proved this result for 1-forms we need to prove it only for a 0-form (differentiable function) g. We need to prove $F^*(dg) = d(F^*(g))$.

For any vector v tangent to M we have

$$\begin{aligned}(F^*(dg))(v) &= dg(F_*v), && \text{by the definition of } F^*,\\ &= F_*v[g], && \text{by the definition of } dg,\\ &= \frac{d}{dt}(g(\beta(t)))\Big|_{t=0},\end{aligned}$$

where β is any curve in N with initial velocity F_*v. By the definition of F_*v we can choose β to be $F(\alpha)$ for some curve α in M with initial velocity v. So

$$\begin{aligned}(F^*(dg))(v) &= \frac{d}{dt}(g\circ F\circ\alpha(t))\Big|_{t=0}\\ &= v[gF], && \text{by the definition of the directional derivative,}\\ &= v[F^*(g)], && \text{by the definition of } F^* \text{ of a 0-form,}\\ &= d(F^*(g))(v), && \text{by the definition of the differential.}\end{aligned}$$

Since this is true for all tangent vectors v it follows that

$$F^*(dg) = d(F^*(g)).$$

IV.7 ORIENTABILITY

Introduction

For this section you need to recall the cross product from Section II.1, the gradient from Section IV.3 and differential forms from Section IV.4.

This last section is rather different from those that have preceded it. We have given a description of surfaces and objects associated with them in terms of local patches. In this section we look at a global property of a surface that can be determined only by examining all of the surface and not just by looking at a single patch. The property of *orientability*, one- or two-sidedness, is defined in terms of differential forms. This shows that these forms are not simply algebraic gadgets but do contain geometric information.

READ: Section IV.7 from page 177 line −6 to the end of page 178.

Comments

(i) *The normal vector field* In Definition 7.4 and Theorem 7.5 we are assuming that the nonvanishing 2-form and the nonvanishing normal vector field are both differentiable. We need to check that the vector field $Z(p) = v \times w/\mu(v, w)$ is differentiable. Assuming that this is defined independently of the choice of v and w we see that Z gives us a normal vector field

$$Z = \frac{x_u \times x_v}{\mu(x_u, x_v)}$$

on the image of a patch x. Since the patch x is differentiable this is a differentiable nonzero normal vector field on x(D). Now any point p in the surface has some neighbourhood that is the image of a patch and hence Z is differentiable on a neighbourhood of p. Hence Z is differentiable on all of the surface.

(ii) *Page 178: line −12, Lemma IV.3.8* Suppose the implicitly defined subset $M : g = c$ is a surface: then, since the converse of Lemma IV.3.8 is not true, it is not necessarily true that ∇g is a nonzero normal vector field and it may not be true that the surface is orientable. We always need to check that dg is nonzero to ensure these results.

Summary

Definition

(i)	Orientable	Page 177, Definition 7.4

Example

(i)	Möbius band	Page 180, Exercise 7

Results

(i)	A surface is orientable if and only if there exists a normal vector field that is nonzero at each point.	Page 178, Theorem 7.5
(ii)	The implicitly defined surface M : g = c is orientable if ∇g is never zero on M.	Page 178, line − 12
(iii)	There exist nonorientable surfaces.	Page 178, line − 9

Technique

(i)	Determining a nowhere zero 2-form on an implicitly defined surface.	Page 178, Theorem 7.5, and Page 178, line − 12

Exercises

Technique (i)

1. Find a nonvanishing 2–form on the sphere $M : x^2 + y^2 + z^2 = c^2$.

Theory Exercise

2. Fill in the details of the proof of Theorem IV.7.5.

Solutions

1. The surface is given by $M : x^2 + y^2 + z^2 = c^2$ and by Lemma IV.3.8 a nonvanishing normal vector field is given by

$$\nabla(x^2 + y^2 + z^2) = 2xU_1 + 2yU_2 + 2zU_3.$$

That is, we can take $Z(p) = 2p$.

Hence by Theorem IV.7.5 a nonvanishing 2-form is given by

$$\begin{aligned}\mu(v, w) &= Z(p)\cdot v \times w\\ &= 2p\cdot v \times w\end{aligned}$$

where v, w are tangent vectors to M at p.

2. (a) Suppose $\mu(v, w) = Z(p)\cdot v \times w$: then to prove that μ is a 2-form we need to check conditions (1) and (2) of Definition IV.4.1. These conditions follow from the analogous properties of the cross product.

(b) If Z(p) is defined by $Z(p) = v \times w/\mu(v, w)$ we need to show that it is independent of the choice of tangent vectors v and w. Suppose $\bar{v}, \bar{w}$ is another pair of linearly independent tangent vectors. Then since the tangent space is of dimension 2 there exist scalars a, b, c, d such that

$$\begin{aligned}\bar{v} &= av + bw\\ \bar{w} &= cv + dw.\end{aligned}$$

Then

$$\overline{Z}(p) = \frac{\overline{v} \times \overline{w}}{\mu(\overline{v}, \overline{w})} = \frac{(av + bw) \times (cv + dw)}{\mu(av + bw, cv + dw)}.$$

Now $(av + bw) \times (cv + dw)$

$$= acv \times v + adv \times w + bcw \times v + bd\, w \times w,$$

by the bilinearity of the cross product,

$$= ad\, v \times w + bc\, w \times v, \qquad \text{since } v \times v = w \times w = \mathbf{0},$$

$$= (ad - bc)(v \times w), \qquad \text{since } w \times v = -v \times w.$$

Now by Lemma IV.4.2

$$\mu(av + bw, cv + dw) = (ad - bc)\, \mu(v, w).$$

Hence

$$\overline{Z}(p) = \frac{(ad - bc)\,(v \times w)}{(ad - bc)\, \mu(v, w)}.$$

Since $\overline{v}$, $\overline{w}$ are linearly independent $ad - bc \neq 0$ and so the above expression is well defined and

$$\overline{Z}(p) = \frac{v \times w}{\mu(v, w)} = Z(p).$$

The vector field is independent of the choice of **v** and **w**.

To see that it is a normal vector field let **u** be any tangent vector to M at **p**. Then there exist scalars a, b such that $u = av + bw$, since the tangent space is 2-dimensional. So

$$\begin{aligned} u \cdot Z(p) &= \frac{(av + bw) \cdot v \times w}{\mu(v, w)} \\ &= \frac{a(v \cdot v \times w) + b(w \cdot v \times w)}{\mu(v, w)} \\ &= 0. \end{aligned}$$

Hence Z(**p**) is perpendicular to each tangent vector to M at **p** and so Z is a normal vector field.

FURTHER EXERCISES AND SOLUTIONS

Section IV.1

Technique Exercises

Pages 131 - 133, Exercises 3, 7, 8, 9, 12.

Other Recommended Exercises

Page 131, Exercise 2.	This deals with descriptions of planes.
Page 132, Exercise 10. Page 133, Exercise 11.	These deal with the images of surfaces under simple mappings.

Solutions

3(a). Page 204, Fig. 5.18 inverted

3(b). Page 204, Fig. 5.19.

3(c). Page 205, Fig. 5.20 straightened out.

3(d). The xy plane.

7. Straightforward.

8. If $(u^2, uv, v^2) = (u'^2, u'v', v'^2)$, with $u', u, v', v > 0$, then $u = u'$ and $v = v'$, so x is one-to-one.

If (p_1, p_2, p_3) is in $x(D)$ then $x^{-1}(p_1, p_2, p_3) = (\sqrt{p_1}, \sqrt{p_3})$. Since the square root function, on the set of positive reals, is continuous it follows that x^{-1} is continuous.

Since the Jacobian matrix of x is

$$\begin{pmatrix} 2u & 0 \\ v & u \\ 0 & 2v \end{pmatrix}$$

which has rank 2 when $u, v > 0$ it follows that x is regular and hence a proper patch.

9. If $u + v = u' + v'$ and $u - v = u' - v'$ then $u = u'$ and $v = v'$. Hence x is one-to-one.

The Jacobian matrix of x is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ v & u \end{pmatrix}$$

which has rank 2 and hence x is regular.

The inverse mapping of $\mathbf{x}$ is

$$\mathbf{x}^{-1}(p_1, p_2, p_3) = \left(\frac{p_1 + p_2}{2}, \frac{p_1 - p_2}{2}\right),$$

which is continuous. Hence $\mathbf{x}$ is a proper patch.

Since $((u + v)^2 - (u - v)^2)/4 = uv$ the image of $\mathbf{x}$ belongs to M. Conversely if (p_1, p_2, p_3) belongs to M then it also belongs to the image of $\mathbf{x}$ since

$$(p_1, p_2, p_3) = (u_0 + v_0, u_0 - v_0, u_0 v_0),$$

where $u_0 = (p_1 + p_2)/2$ and $v_0 = (p_1 - p_2)/2$. Hence M is the image of $\mathbf{x}$.

12. When we revolve we preserve the x coordinate and the square of the distance from the x axis. Hence M is the surface $M : f(x, y^2 + z^2) = c = g(x, y, z)$. Using the chain rule and the fact that C is a curve we can show that dg is never zero on M and hence M is a surface.

2. $M : (a, b, c) \cdot \left((x, y, z) - \frac{d(a, b, c)}{a^2 + b^2 + c^2}\right) = 0.$

10. If $\mathbf{p}$ is a point of F(M) then $\mathbf{p} = F(\mathbf{q})$ for some point $\mathbf{q}$ in M. Since M is a surface there is a proper patch $\mathbf{x}$ giving a neighbourhood of $\mathbf{q}$ in M. We must show that $F(\mathbf{x})$ is a proper patch containing $\mathbf{p}$ in F(M).

Since $\mathbf{x}$ is one-to-one and F, an isometry, is also one-to-one it follows that $F(\mathbf{x})$ is one-to-one. Since $\mathbf{x}$ is regular, F is an isometry and $(F(\mathbf{x}))_* = F_* \mathbf{x}_*$ it follows that $(F(\mathbf{x}))_*$ is one-to-one and hence $F(\mathbf{x})$ is regular. Since $\mathbf{x}$ is a proper patch, F is an isometry and $(F(\mathbf{x}))^{-1} = \mathbf{x}^{-1} F^{-1}$ it follows that $(F(\mathbf{x}))^{-1}$ is continuous and hence $F(\mathbf{x})$ is a proper patch. So F(M) is a surface.

11.
$$\begin{aligned} d\bar{g}(F_* v) &= (F_* v)[\bar{g}] \\ &= v[\bar{g}(F)], \quad \text{by Exercise I.7.9,} \\ &= v[g \circ F^{-1} \circ F] \\ &= v[g] \\ &= dg(v) \neq 0. \end{aligned}$$

Section IV.2

Technique Exercises

Pages 140-143, Exercises 3, 5, 8, 9, 10, 11.

Page 143, Exercise 13. This deals with simplification of parametrizations of surfaces of revolution.

Solutions

3. A profile curve is $C : (x - 4)^2 + z^2 = 4$, in the xz plane. The axis of revolution is the z axis.

5. $\mathbf{x}_u \times \mathbf{x}_v = v\delta' \times \delta$

This is nonzero, and hence $\mathbf{x}$ is regular, if and only if v and $\delta \times \delta'$ are nonzero.

8(a).

$$\mathbf{x}_u \times \mathbf{x}_v = \begin{vmatrix} U_1 & U_2 & U_3 \\ \cos v & 1 & 0 \\ -u \sin v & 0 & \cos v \end{vmatrix}$$

$$= \cos v \, U_1 - \cos^2 v \, U_2 + u \sin v \, U_3$$

$$\neq 0 \text{ if } u > 0.$$

8(b). $M : y - (x^2 + y^2 z^2)^{\frac{1}{2}} = 0.$

8(c). $\mathbf{x}(u, v) = (0, 0, \sin v) + u(\cos v, 1, 0).$

9(a). The direction of L is $\beta'(u) + U_3$.

Hence $\mathbf{x}(u, v) = \beta(u) + v(\beta'(u) + U_3)$.

9(b). $\beta(u) = (\cos u, \sin u, 0)$.

$\mathbf{x}(u, v) = (\cos u - v \sin u, \sin u + v \cos u, v)$.

9(c). The above points belong to $M' : x^2 + y^2 - z^2 = 1$. On M' we have $x^2 + y^2 = 1 + z^2$ and so we can write

$$x = (1 + v^2)^{\frac{1}{2}} \cos t$$

$$y = (1 + v^2)^{\frac{1}{2}} \sin t, \text{ where } z = v.$$

Let α be such that $\cos \alpha = (1 + v^2)^{-\frac{1}{2}}$, $\sin \alpha = v(1 + v^2)^{-\frac{1}{2}}$.

Let $t = u + \alpha$: then

$$x = \cos u - v \sin u$$

$$y = \sin u + v \cos u.$$

Hence M is the implicitly defined surface M'.

9(d). $\mathbf{x}(u, v) = (\cos u - v \sin u, \sin u + v \cos u, - v)$. $M : x^2 + y^2 - z^2 = 1$.

9(e). See Fig. 5.38, page 232.

10(a). $dg = \frac{2x}{a^2} dx + \frac{2y}{b^2} dy + \frac{2z}{c^2} dz \neq 0$ on M.

$\mathbf{x}_u \times \mathbf{x}_v = (-bc \cos^2 u \cos v, - ac \cos^2 u \sin v, - ab \sin u \cos u)$.

So if $-\frac{\pi}{2} < u < \frac{\pi}{2}$ then $\cos u$ is never zero and $\mathbf{x}_u \times \mathbf{x}_v$ is never zero.

The image of $\mathbf{x}$ is the whole of M apart from the two points $(0, 0, \pm c)$.

10(b). $dg = \frac{2x}{a^2} dx + \frac{2y}{b^2} dy - \frac{2z}{c^2} dz \neq 0$ on M.

$\mathbf{x}_u \times \mathbf{x}_v = (-bc \cosh^2 u \cos v, - ac \cosh^2 u \sin v, ab \cosh u \sinh u)$, which is never zero, as $\cosh u$ is never zero.

The image of $\mathbf{x}$ is the whole of M.

10(c). $dg = \frac{2x}{a^2} dx + \frac{2y}{b^2} dy - \frac{2z}{c^2} dz \neq 0$ on M.

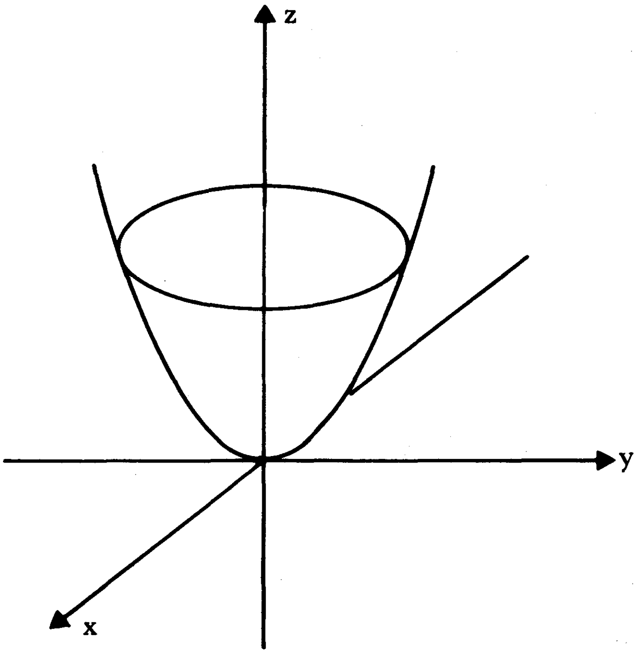
$\mathbf{x}_u \times \mathbf{x}_v = (-bc \sinh^2 u \cos v, - ac \sinh^2 u \sin v, ab \cosh u \sinh u)$, which is never zero, since $\sinh u$ is never zero $(u \neq 0)$.

The image of $\mathbf{x}$ is the whole of M apart from the two points $(0, 0, \pm c)$.

11(a). $dg = \frac{2x}{a^2}dx + \frac{2y}{b^2}dy - dz \neq 0.$

$x_u \times x_v = (-2bu^2\cos v, -2au^2\sin v, abu)$, which is never zero, since $u > 0$. The parametrization omits only the point (0, 0, 0).

11(b). The u-parameter curves $v = v_0$ are parabolas in vertical planes through the z axis. The v-parameter curves $u = u_0$ are horizontal ellipses about the z axis.



13(a). If g' is never zero it has an inverse function k, defined on some suitable interval. The curve can be reparametrized as

$$u \longmapsto (g(k(u)), h(k(u)), 0) = (u, f(u), 0).$$

This gives the required parametrization of M.

13(b). As for (a).

Section IV.3

Technique Exercises

Pages 150 - 151, Exercises 7, 8(b), 9, 10.

Solutions

7. $M : xy - z = 0 = g(x, y, z)$.

A normal vector field is $\nabla g = yU_1 + xU_2 - U_3$.

The surface is given by the patch $x(u, v) = (u, v, uv)$.

Linearly independent tangent vector fields are

$$x_u = (1, 0, v) = (1, 0, y)$$
$$x_v = (0, 1, u) = (0, 1, x).$$

8(b). $\alpha' = \sqrt{2}\, x_u + e^t x_v$

$$\alpha' \cdot x_u/\|x_u\| = \sqrt{2}\|x_u\| = \sqrt{2}\|e^t(-\sin\sqrt{2}t, \cos\sqrt{2}t, 0)\|$$
$$= \sqrt{2}e^t$$
$$\alpha' \cdot x_v/\|x_v\| = e^t\|x_v\| = e^t\,\|(\cos\sqrt{2}t, \sin\sqrt{2}t, 1)\|$$
$$= \sqrt{2}e^t.$$

9(a). The plane $\overline{T}_p(M)$ consists of all those points q such that

$$(q - p)\cdot z = 0.$$

If v is a tangent to M at p then p + v belongs to $\overline{T}_p(M)$ since

$$(p + v - p)\cdot z = v \cdot z = 0.$$

Conversely if q belongs to $\overline{T}_p(M)$ then since $(q - p)\cdot z = 0$ the vector $v = q - p$ is a tangent vector. Hence $q = p + v$ for some tangent vector v.

9(b). $x_u \times x_v$ is a never zero normal vector field.

9(c). ∇g is a never zero normal vector field if dg is never zero.

10(a). $(r - p)\cdot\nabla g(p) = 0$ gives the plane $z = 0$.

10(b). $((x, y, z) - (1, -2, 3))\cdot(\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}) = \frac{x}{2} - \frac{y}{4} + \frac{z}{3} - 2 = 0.$

10(c). $((x, y, z) - (\sqrt{2}, \sqrt{2}, \frac{\pi}{2}))\cdot(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) \times (-\sqrt{2}, \sqrt{2}, 2) = 0$

$$((x, y, z) - (\sqrt{2}, \sqrt{2}, \tfrac{\pi}{2}))\cdot(\sqrt{2}, -\sqrt{2}, 2) = 0$$

$$\sqrt{2}x - \sqrt{2}y + 2z = \pi.$$

Section IV.4

Technique Exercises

Pages 157 - 158, Exercises 3, 5, 6, 9, 10.

Solutions

3.
$$\begin{aligned} d(fg)(v_p) &= v_p[fg] = v_p[f]\ g(p) + f(p)\,v_p[g] \\ &= df(v_p)g(p) + f(p)dg(v_p) \\ &= (df\, g + f\, dg)(v_p). \end{aligned}$$

$$\begin{aligned} d(f\phi)(x_u, x_v) &= \frac{\partial}{\partial u}(f\phi(x_v)) - \frac{\partial}{\partial v}(f\phi(x_u)) \\ &= \frac{\partial}{\partial u}(f(x)\phi(x_v)) - \frac{\partial}{\partial v}(f(x)\phi(x_u)) \\ &= f(x)\,(\frac{\partial}{\partial u}(\phi(x_v)) - \frac{\partial}{\partial v}(\phi(x_u))) \\ &\quad + \frac{\partial(f(x))}{\partial u}\phi(x_v) - \frac{\partial(f(x))}{\partial v}\phi(x_u) \\ &= f(x)\, d\phi(x_u, x_v) \\ &\quad + df(x_u)\phi(x_v) - df(x_v)\phi(x_u) \\ &= (df \wedge \phi + fd\phi)\,(x_u, x_v). \end{aligned}$$

5. By analogy with Lemma I.5.7

$$v_p[g(f)] = g'(f)(p)v_p[f]$$

and hence

$$d(g(f))(v_p) = (g'(f)df)(v_p).$$

6(a). $d(fgh) = df\,gh + f\,dgh$

$$= df\,gh + f\,dg\,h + fg\,dh.$$

6(b). $d(\phi f) = d(f\phi) = df \wedge \phi + fd\phi = fd\phi - \phi \wedge df.$

6(c). $(df \wedge dg)(v, w) = df(v)dg(w) - df(w)dg(v)$

$$= v[f]\,w[g] - v[g]\,w[f].$$

9(a). If $\pi_1 : E^2 \longrightarrow R$ is given by

$$\pi_1(u, v) = u$$

then $\tilde{u} = \pi_1 x^{-1}$.

This is differentiable on $x(D)$ if the coordinate expression

$$\tilde{u}y = \pi_1 x^{-1} y$$

is differentiable for all patches y. It is, because π_1 and $x^{-1}y$ are both differentiable.

Also $\tilde{u}(x(u, v)) = \pi_1 x^{-1} x(u, v) = \pi_1(u, v) = u.$

9(b). $d\tilde{u}(x_u) = x_u[\tilde{u}] = \frac{\partial}{\partial u}(\tilde{u}x) = \frac{\partial}{\partial u}(\pi_1) = 1$, etc.

9(c). $\phi(x_u) = (\phi(x_u)d\tilde{u} + \phi(x_v)d\tilde{v})(x_u)$, etc,

and

$$\eta(x_u, x_v) = (\eta(x_u, x_v)d\tilde{u}d\tilde{v})(x_u, x_v),$$

since

$$d\tilde{u}d\tilde{v}(x_u, x_v) = d\tilde{u}(x_u)d\tilde{v}(x_v) - d\tilde{u}(x_v)d\tilde{v}(x_u) = 1.$$

10(a). The coordinate system gives the radius and the angle the radius makes with the x axis.

10(b). The coordinate system is the natural coordinate system.

10(c). The coordinate system gives the longitude and latitude.

Section IV.5

Technique Exercises

Page 165, Exercises 6 and 7.

Other Recommended Exercises

Page 165, Exercise 4. Read Exercise 3 and use it to construct a mapping of the helicoid to the torus.

Page 166, Exercise 11. Read Exercise 10 and use it to construct mappings of the torus to itself.

Page 166, Exercise 14(c). This deals with the effect of composite mappings on forms.

Page 166, Exercise 16. This deals with stereographic projection.

Solutions

6(a). Let U_1, U_2 be the natural frame field on E^2.

Then

$$(x^*(\phi))(U_1) = \phi(x_*U_1) = \phi(x_u) \text{ and}$$
$$(x^*(\phi))(U_2) = \phi(x_v).$$

6(b). $(x^*(\upsilon))(U_1, U_2) = \upsilon(x_*U_1, x_*U_2) = \upsilon(x_u, x_v)$

6(c). $(x^*(d\phi))(U_1, U_2) = d\phi(x_u, x_v)$

$$= \left(\frac{\partial}{\partial u}(\phi x_v) - \frac{\partial}{\partial v}(\phi x_u)\right).$$

7(a).

$$\begin{aligned} x(u, v) &= (r \cos v \cos u,\ r \cos v \sin u,\ r \sin v) \\ x_u &= (-r \cos v \sin u,\ r \cos v \cos u,\ 0) \\ x_v &= (-r \sin v \cos u,\ -r \sin v \sin u,\ r \cos v). \end{aligned}$$

Hence

$$\phi(x_u) = r^2 \cos^2 v,\ \phi(x_v) = 0$$

and

$$\begin{aligned} x^*(\phi) &= r^2 \cos^2 v\ du \\ x^*(d\phi) &= 2r^2 \cos v \sin v\ \ du\, dv. \end{aligned}$$

7(b). $x^*(\upsilon) = r^3 \cos v \sin^2 v\ \ du\, dv.$

4. $(u \cos v,\ u \sin v,\ bv)$

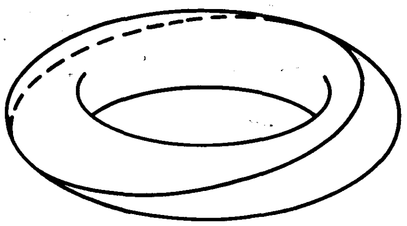
$\longmapsto ((R + r \cos u)\cos v,\ (R + r \cos u)\sin v,\ r \sin u).$

11(a). This maps the meridians three times round themselves. Not a diffeomorphism.

11(b). This rotates each meridian circle by 180°. A diffeomorphism.

11(c). This interchanges parallels and meridians. A diffeomorphism.

11(d). This maps meridians and parallels to curves that wrap round the torus as in the following diagram. Not a diffeomorphism.



14(c). $(GF)^*(f) = f(GF) = F^*(fG) = F^*G^*(f).$

$$((GF)^*\phi)(v) = \phi((GF)_*v) = \phi(G_*(F_*(v)))$$
$$= (G^*\phi)(F_*(v)) = ((F^*G^*)\phi)(v).$$
$$((GF)^*\eta)(v, w) = \eta((GF)_*v, (GF)_*w) = \eta(G_*(F_*(v)), G_*(F_*(w)))$$
$$= (G^*\eta)(F_*v, F_*w) = ((F^*G^*)\eta)(v, w).$$

16(a). Straightforward.

16(b). P^{-1} gives a patch covering all of the sphere apart from one point. Another similar patch completes the covering of the sphere.

Section IV.7

Technique Exercises

Pages 180 - 181, Exercises 7, 8 (Is M^* orientable?).

Erratum: The last line on page 180 should read

$$"x(u, v) = \beta(u) + v\delta(u), -\tfrac{1}{2} < v < \tfrac{1}{2},".$$

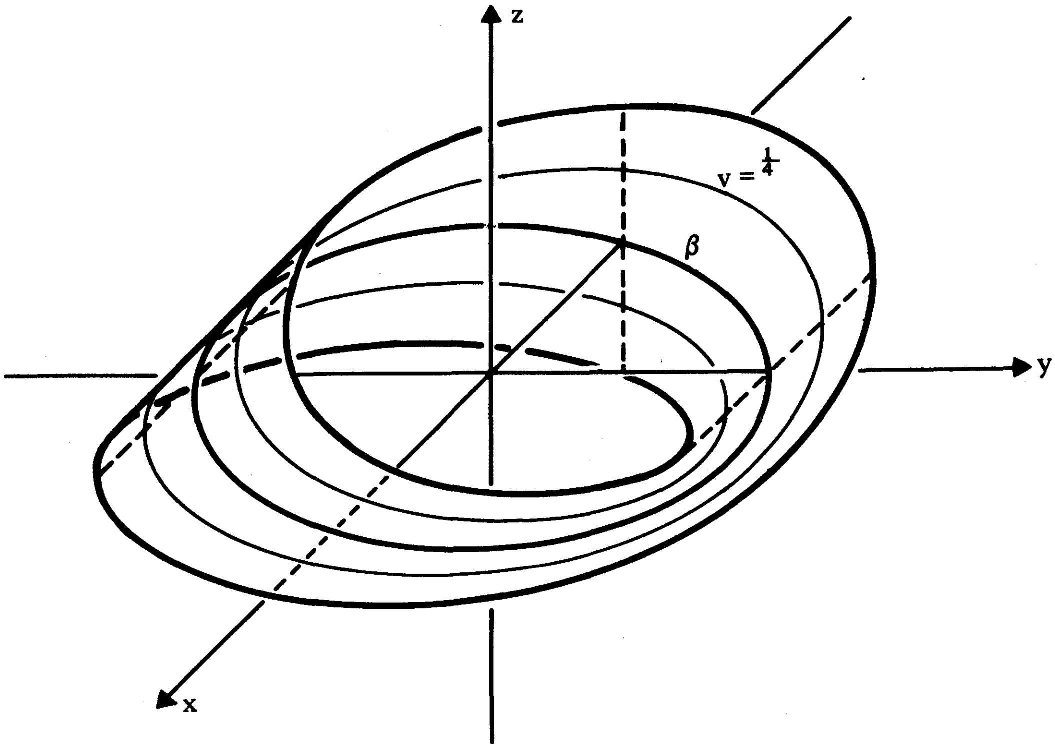
Other Recommended Exercises

Pages 180 - 181, Exercises 3, 9(b) and 9(c). These relate orientability to mappings.

Solutions

7(a). E, F, G were defined on page 140.

7(b).



The u-parameter curve $v = v_0 \neq 0$ is given by

$$u \longmapsto \beta(u) + v_0\delta(u).$$

Since β has period 2π and δ has period 4π this curve has period 4π.

8. M^* is the image, under x, of the region $0 < v$. In this region two points (u, v) and (u', v') give the same image under x if and only if $v = v'$ and $u = u' + 4\pi n$, for some integer n. The same is true of the vectors $x_u(u, v) \times x_v(u, v)$ and $x_u(u', v') \times x_v(u', v')$.

   Hence $x_u \times x_v$ defines a normal vector field on M^*.

3. If μ is a never zero 2-form on N we prove that $F^*(\mu)$ is a never zero 2-form on M since

$$F^*(\mu)(x_u, x_v) = \mu(F_*(x_u), F_*(x_v)) \neq 0$$

   since F is regular.

9(b). Let F map the Möbius band M (see Exercise 7) to the cylinder $N : x^2 + y^2 = 1$ as follows:

$$F : x(u, v) \longmapsto \beta(u) = (\cos u, \sin u, 0).$$

9(c). We want a regular mapping $F : M \longrightarrow N$ where M is orientable and N is not. Choose any patch on the Möbius band. For instance take $x : D \longrightarrow M$ where x is as in Exercise 7 and D is the set of points in E^2 such that $-\pi < u < \pi$, and $-\frac{1}{2} < v < \frac{1}{2}$. We know that E^2 can be considered as a surface in E^3.

DIFFERENTIAL GEOMETRY

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II Frame Fields

III Euclidean Geometry

IV Calculus on a Surface

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VI Geometry of Surfaces in E^3

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Set Book

Barrett O'Neill: *Elementary Differential Geometry* (Academic Press, 1966). It is essential to have this book: the course is based on it and will not make sense without it.

Bibliography

The set books for M201, M231 and MST 282 are referred to occasionally; they are useful but not essential. They are:

D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins: *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

E.D. Nering: *Linear Algebra and Matrix Theory* (John Wiley, 1970).

M. Spivak: *Calculus,* paperback edition (W. A. Benjamin/Addison-Wesley, 1973).

R.C. Smith and P. Smith: *Mechanics,* SI edition (John Wiley, 1972).

Conventions

Before starting work on this text, please read M334 Part Zero. Consult the Errata List and the Stop Press and make any necessary alterations for this chapter in the set book.

Unreferenced pages and sections denote the set book. Otherwise

O'Neill	denotes the set book;
Text	denotes the correspondence text;
KKOP	denotes *An Introduction to Linear Analysis* by D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins;
Nering	denotes *Linear Algebra and Matrix Theory by* E.D. Nering;
Spivak	denotes *Calculus* by M. Spivak;
Smith	denotes *Mechanics* by R.C. Smith and P. Smith.

Reference to Open University Courses in Mathematics take the form:

Unit M100 22, Linear Algebra I

Unit MST 281 10, Taylor Approximation

Unit M201 16, Euclidean Spaces I: Inner Products

Unit M231 2, Functions and Graphs

Unit MST 282 1, Some Basic Tools.

V.1 INTRODUCTION AND THE SHAPE OPERATOR OF $M \subset E^3$

Introduction

This section depends upon a knowledge of *curves in a surface* (Section IV.3) and *covariant derivatives* (Section II.5). We need no concept from Chapter IV from any sections later than Section IV.3 except that of *orientability* (Section IV.7) and that only so that we may use "orientable surface" as a shorthand for "surface which has a unit normal vector field defined on it". We need the fact that all surfaces are locally orientable; that is, given any point p in a surface M there is some neighbourhood N of p in M which is orientable. This is because there is some neighbourhood of p which lies in the image of a patch x, and this is oriented by the unit normal vector field $U = (x_u \times x_v)/\|x_u \times x_v\|$.

Throughout this chapter we confine our attention to surfaces M which are *connected* in the sense of O'Neill, Section IV.7; that is, given any points p, q in M there is a curve $\alpha: I \longrightarrow M$, where $[0, 1] \subset I$, such that $\alpha(0) = p$, $\alpha(1) = q$. This definition of "connected" is in general different from that given in *Unit M202 27, Connectedness* and *Unit M332 2, Continuous Functions* (in which M is connected if and only if it cannot be expressed as the disjoint union of two non-empty open subsets of M), but it may be shown that for surfaces in E^3 the two definitions are equivalent (where, for the M202 definition, the topology on M is that of a subspace of R^3 with the usual topology). Thus if you have done M202 or M332 you may use whichever definition of "connected" that you find easier.

In Chapter IV we established techniques for describing and investigating surfaces in E^3. Now we can use these techniques to look at some geometric properties of surfaces in E^3. One such property is the *shape* of a surface: we can describe the shape of a surface M near a point p in M by measuring the rate of change of a unit normal vector field to M along curves in M through p. We do this by using the covariant derivative. We shall need to modify somewhat our previous definition of covariant derivative so that we can apply it to vector fields defined on surfaces only. In Section II.5 $\nabla_v Z$ was defined as

(i) $\nabla_v Z = Z(p + tv)'(0)$:

in Exercise II.5.6 and Lemma II.5.2 it was shown that this definition is equivalent to

(ii) $\nabla_v Z = (Z \circ \alpha)'(0)$ for any curve α with $\alpha'(0) = v$

and

(iii) $\nabla_v Z = \Sigma v[z_i]\ U_i(p)$ where z_i are the coordinate functions of Z.

If we restrict Z to be a vector field defined on a surface M, definition (i) is no longer available to us, as the curve $t \longmapsto p + tv$ may not lie within M. However, if we also restrict v to be a tangent vector to E^3 which is tangent to M then by definition there is some curve α in M with $\alpha'(0) = v$: now everything in definition (ii) is meaningful, and accordingly we take the following as our new definition.

Definition If Z is a vector field on a surface M and v is a vector tangent to M, then the covariant derivative $\nabla_v Z$ is the common value of $(Z \circ \alpha)'(0)$ for all curves α in M with $\alpha'(0) = v$.

As before, this is equivalent to

$$\begin{aligned} \nabla_v Z &= \Sigma v[z_i]\ U_i(p) \\ &= \Sigma (z_i \circ \alpha)'(0)\ U_i(p) \end{aligned}$$

for all curves α in M with $\alpha'(0) = v$ (by Definition IV.3.10), and the arguments in

Comment (v) to Section IV.3 (Text, page 35) which show that $v[z_i]$ is independent of the choice of α can be extended to show that $\nabla_v Z$ is also independent of the choice of α. Moreover, if Z is the restriction to M of a vector field W on E^3 (equivalently, if z_i is the restriction to M of a differentiable function w_i on E^3 for $i = 1, 2, 3$) then the covariant derivative $\nabla_v Z$ obtained from the above definition is the same as the covariant derivative $\nabla_v W$ obtained as in Section II.5: thus we may evaluate the covariant derivative by any method from Section II.5 in such cases.

We spend the first three sections of this chapter defining, and giving geometric meaning to, concepts such as *shape, normal curvature, Gaussian curvature,* and defer until Section V.4 explicit methods for calculating these for general surfaces.

READ: Introduction to Chapter V and Section V.1 (pages 189-193).

Comments

(i) *Page 190: lines 14-16* We have not proved that there are exactly two unit normal vector fields on a connected orientable surface, as the proof requires more work on connectedness outside the main stream of this course. However, the idea of the proof is very simple: if Z is a unit normal vector field then at each point Z must be either U or −U, and on a connected surface Z cannot suddenly flip over from U to −U without losing its differentiability.

(ii) *Page 192: line 1* The vector field used here and in Example IV.3.9, $X = \Sigma x_i U_i$, is an interesting one that will be used later in the chapter. It has the property that

$$\nabla_v X = \Sigma v[x_i]\ U_i(p) = \Sigma v_i U_i(p) = v$$

and similarly,

$$\nabla_V X = V.$$

(iii) *Page 193: the last paragraph* The name "symmetric linear transformation" derives from the fact that the matrix of a symmetric linear transformation with respect to any orthonormal basis is a symmetric matrix. In the two-dimensional case this means that with respect to an orthonormal basis the matrix has the form

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where $a_{12} = a_{21}$. The significance of such transformations is that they are always diagonalizable; that is, there is some orthonormal basis with respect to which the matrix is diagonal. Such a basis consists of eigenvectors of the linear transformation. For example, in the two-dimensional case the matrix with respect to an eigenvector basis has the form

$$B = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

where k_1 and k_2 are eigenvalues of the transformation. The eigenvalues k_1 and k_2 may possibly be equal, and either of them may be zero. O'Neill uses the words *characteristic vector, characteristic value* to refer to eigenvectors and eigenvalues.

O'Neill also refers to the *determinant* and *trace* of a linear transformation. You have met determinants already in this course. The determinant of a linear transformation T from a finite-dimensional vector space V into itself is defined to be the determinant of any matrix which represents the transformation T; it can be shown that this is independent of the choice of basis. In the above examples

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

and

$$\det B = k_1k_2.$$

The trace of a matrix is defined to be the sum of its diagonal terms. Thus

$$\text{trace } A = a_{11} + a_{22}$$

and

$$\text{trace } B = k_1 + k_2.$$

The trace of a linear transformation T from a finite-dimensional vector space V into itself is defined to be the trace of the matrix of T with respect to any basis of V: again it can be shown that this is independent of the choice of basis. For example, the trace of the identity mapping from E^n to E^n is simply n.

Additional Text

We shall need the result of the following exercise throughout the remainder of Chapter V.

1. Page 194, Exercise 1. (HINT: Use the result of Exercise II.5.6(a).)

Summary

Notation

$\nabla_v Z$	Page 189, line -6
S_p	Page 190, Definition 1.1
S	Page 191, line -8
f_x, f_y, f_{xx}, f_{xy}, f_{yy}	Text: page 8, Exercise 4

Definitions

(i)	The covariant derivative $\nabla_v Z$ on a surface	Text, page 5
(ii)	The shape operator at p, S_p	Page 190, Definition 1.1
(iii)	The shape operator S	Page 191, line -9
(iv)	The trace of a linear operator	Text, page 7

Examples

(i)	If M is a sphere of radius r, then $S = -\frac{1}{r}I.$	Pages 191-192, Example 1.3(1)
(ii)	If M is a circular cylinder of radius r, then $S(v) = 0$ if v is tangent to a ruling of M and $S(v) = -v/r$ if v is tangent to a cross-sectional circle of M.	Page 192, Example 1.3(3)

Results

(i)	S_p is a symmetric linear transformation from $T_p(M)$ to $T_p(M)$.	Page 191, Lemma 1.2 and Page 193, Lemma 1.4
(ii)	$S(\alpha') = -U'$, where U is restricted to α.	Page 194, Exercise 1
(iii)	If $X = \Sigma x_i U_i$, then $\nabla_v X = v$.	Text, page 6

Techniques

(i)	Calculating the covariant derivative of a vector field on a surface using $\left.\begin{array}{ll}\text{(a)} & \nabla_v Z = (Z \circ \alpha)'(0) \\ \text{(b)} & \nabla_v Z = \Sigma\,(z_i \circ \alpha)'(0) U_i(p)\end{array}\right\}$ for $\alpha'(0) = v_p$.	Page 189, Method 1 Page 190, Method 2
(ii)	Where Z, z_i are restrictions to M of a vector field on E^3 and differentiable functions on E^3, calculating the covariant derivative using	
	(a) $\nabla_v Z = \Sigma\, z_i(p + tv)'(0)\ U_i(p)$	Page 78, Lemma II.5.2
	(b) $\nabla_v Z = \Sigma v[z_i]\, U_i(p)$, where $v[z_i]$ is evaluated using $U_i(p)[x_j] = \delta_{ij}$ and the linear and Leibnizian properties.	Page 80, Corollary II.5.4
(iii)	Finding the matrix representing S_p with respect to some basis for $T_p(M)$, and hence determining the rank of S_p (that is, the number of linearly independent rows) and its effect on the whole of $T_p(M)$.	Page 190, Definition 1.1

Exercises

Technique (i)

2. If $x : D \longrightarrow M$ is a patch, Z a vector field on M, and w is the tangent vector $x_u(u_0, v_0)$ to M at $x(u_0, v_0)$, show that

$$\nabla_w Z = \frac{\partial}{\partial u}(Z(x))(u_0, v_0) = \sum_i \frac{\partial z_i(x)}{\partial u}(u_0, v_0)\ U_i\ (x(u_0, v_0)).$$

Technique (ii)(a)

3. Calculate $\nabla_v Z$ for $v = (0, 1, 0)$ at the point $(r, 0, 0)$ on the cylinder $x^2 + y^2 = r^2$ when $Z = -yU_1 + xU_2$.

Technique (ii)(b)

4. Page 194, Exercise 2. What is the matrix of S_0 with respect to $\{u_1, u_2\}$?. We shall use the result of this exercise in the next section, but it is not worth remembering for its own sake. The symbol f_x denotes $\partial f/\partial x$, $f_y = \partial f/\partial y$, $f_{xx} = \partial^2 f/\partial x^2$, $f_{xy} = \partial^2 f/\partial x \partial y$, $f_{yy} = \partial^2 f/\partial y^2$.

Technique (iii)

5. Page 194, Exercise 3. Give the matrix of S_0 with respect to $\{u_1, u_2\}$ in each case.

Optional Section – The Gauss Mapping

Historically the unit normal vector field U was studied by investigating the Gauss mapping. Although this is outside the scope of this course, the Gauss mapping does play an important role in differential geometry: for example, it is used in the second of the sections in O'Neill on Integration (VI.7), which are omitted from M334 but which you may like to read later. If you would like to study the Gauss mapping, read the definition given in Exercise 4 on page 194 and then do the following exercises.

6. Page 194, Exercise 4.

7. Page 195, Exercise 5.

8. Page 195, Exercise 6.

9. Page 195, Exercise 7.

Solutions

1. Page 194, Exercise 1.

We need to show that

$$S(\alpha'(t_0)) = -(U(\alpha))'(t_0)$$

for all t_0 in the domain of α. Putting $v = \alpha'(t_0)$,

$$\begin{aligned} S(\alpha'(t_0)) &= S(v) \\ &= -\nabla_v U, && \text{by Definition V.1.1,} \\ &= -(U \circ \beta)'(0), && \text{by definition of the covariant derivative,} \end{aligned}$$

where β is any curve in M with $\beta'(0) = \alpha'(t_0)$. An obvious choice for β is

$$\beta : t \longmapsto \alpha(t + t_0).$$

Now

$$U(\beta(t)) = U(\alpha(t + t_0))$$

so

$$(U(\beta))'(t) = (U(\alpha))'(t + t_0)$$

so

$$(U(\beta))'(0) = (U(\alpha))'(t_0).$$

Thus

$$S(\alpha'(t_0)) = -(U(\beta))'(0) = -(U(\alpha))'(t_0).$$

Since we are considering just the restriction of U to the curve α, so that we are thinking of U as a vector field on a curve, we may write U' for $(U(\alpha))'$. Thus the last equation becomes, on omitting t_0,

$$S(\alpha') = -U'.$$

2. The vector $w = x_u(u_0, v_0)$ is the tangent vector to the curve $u \longmapsto x(u, v_0)$ at the point $x(u_0, v_0)$, so if we take α to be the curve

$$\alpha : t \longmapsto x(u_0 + t, v_0)$$

lying in M, then $x_u(u_0, v_0) = \alpha'(0)$. We use Method 1 on pages 189-190 to obtain

$$\begin{aligned} \nabla_w Z &= (Z(\alpha))'(0) \\ &= \frac{d}{dt}(Z(\alpha(t)))\Big|_{t=0} \\ &= \frac{d}{dt}(Z(x(u_0 + t, v_0)))\Big|_{t=0} \\ &= \frac{\partial}{\partial u}(Z(x))(u_0, v_0) \quad \text{by definition of partial differentiation.} \end{aligned}$$

Similarly, $(z_i(\alpha))'(0) = \frac{\partial}{\partial u}(z_i(x))(u_0, v_0)$, and so Method 2 gives

$$\nabla_w Z = \Sigma\,(z_i(\alpha))'(0)\; U_i\,(x(u_0, v_0)) = \Sigma \frac{\partial}{\partial u}(z_i(x))(u_0, v_0)\; U_i\,(x(u_0, v_0)).$$

3. $\nabla_v(-yU_1 + xU_2) = (-y(p + tv))'(0)\, U_1(p) + (x(p + tv))'(0)\, U_2(p)$.

In this case

$$\begin{aligned} p + tv &= (r, 0, 0) + t(0, 1, 0) \\ &= (r, t, 0) \end{aligned}$$

so

$$-y(p + tv) = -t$$

so

$$(-y(p + tv))'(0) = -1,$$

and

$$x(p + tv) = r$$

so

$$(x(p + tv))'(0) = 0.$$

Thus

$$\nabla_v(-yU_1 + xU_2) = -U_1(p) = -U_1(r, 0, 0).$$

4. Page 194, Exercise 2.

(a) Since M is given implicitly as $M : g = 0$, where

$$g(x, y, z) = z - f(x, y),$$

we know from Lemma IV.3.8 that ∇g is a normal vector field on M.

$$\begin{aligned} \nabla g &= \frac{\partial g}{\partial x} U_1 + \frac{\partial g}{\partial y} U_2 + \frac{\partial g}{\partial z} U_3 \\ &= -f_x U_1 - f_y U_2 + U_3. \end{aligned}$$

This has norm $(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}$, and so a unit normal vector field to M is given by

$$U = \frac{-f_x U_1 - f_y U_2 + U_3}{(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}}.$$

Now $U(0, 0, 0) = (0, 0, 1)$ because $f_x(0, 0) = f_y(0, 0) = 0$. Tangent vectors at $(0, 0, 0)$ are tangent to M if and only if they are orthogonal to $U(0, 0, 0) = (0, 0, 1)$. The tangent vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are both orthogonal to $(0, 0, 1)$, and so they are tangent to M at the origin.

(b) By Method 2 on page 190,

$$\nabla_{\mathbf{u}_1} U = \mathbf{u}_1\left[\frac{-f_x}{(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}}\right] U_1(0) + \mathbf{u}_1\left[\frac{-f_y}{(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}}\right] U_2(0)$$

$$+ \mathbf{u}_1\left[\frac{1}{(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}}\right] U_3(0).$$

We know that S_p maps $T_p(M)$ to $T_p(M)$ and so the component of $\nabla_{\mathbf{u}_1} U$ in the $U_3(0)$ direction is zero.

Now $\mathbf{u}_1[h] = (\partial h/\partial x)(0, 0, 0)$ for any function h. Since we know that $f_x(0, 0) = f_y(0, 0) = 0$ we can ignore any terms having f_x or f_y as a factor when we have differentiated, so

$$\mathbf{u}_1\left[\frac{-f_x}{(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}}\right] = \frac{\partial}{\partial x}\left(\frac{-f_x}{(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}}\right)(0, 0, 0)$$

$$= \frac{-f_{xx}}{(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}}(0, 0, 0)$$

$$= -f_{xx}(0, 0);$$

Similarly

$$\mathbf{u}_1\left[\frac{-f_y}{(f_x{}^2 + f_y{}^2 + 1)^{\frac{1}{2}}}\right] = -f_{xy}(0, 0)$$

Thus

$$\nabla_{\mathbf{u}_1} U = -f_{xx}(0, 0)\, U_1(0) - f_{xy}(0, 0)\, U_2(0).$$

Now

$$\begin{aligned} S(\mathbf{u}_1) &= -\nabla_{\mathbf{u}_1} U \\ &= f_{xx}(0, 0)\, U_1(0) + f_{xy}(0, 0)\, U_2(0) \\ &= f_{xx}(0, 0)\mathbf{u}_1 + f_{xy}(0, 0)\mathbf{u}_2. \end{aligned}$$

In an exactly similar manner, since $\mathbf{u}_2[h] = (\partial h/\partial y)(0, 0, 0)$, we can show that

$$S(\mathbf{u}_2) = f_{xy}(0, 0)\mathbf{u}_1 + f_{yy}(0, 0)\mathbf{u}_2.$$

The tangent vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ form a basis for $T_0(M)$. The two equations

$$S(\mathbf{u}_1) = f_{xx}(0, 0)\mathbf{u}_1 + f_{xy}(0, 0)\mathbf{u}_2$$
$$S(\mathbf{u}_2) = f_{xy}(0, 0)\mathbf{u}_1 + f_{yy}(0, 0)\mathbf{u}_2$$

tell us that the matrix of S with respect to this basis is

$$\begin{pmatrix} f_{xx}(0, 0) & f_{xy}(0, 0) \\ f_{xy}(0, 0) & f_{yy}(0, 0) \end{pmatrix}$$

5. Page 194, Exercise 3.

In each case we can put $z = f(x, y)$ and show that

$$f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0.$$

Then the result of the previous exercise tells us that, with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$, the matrix of $\mathbf{S_0}$ is

$$\begin{pmatrix} f_{xx}(0, 0) & f_{xy}(0, 0) \\ f_{xy}(0, 0) & f_{yy}(0, 0) \end{pmatrix}$$

Expressing every vector in $T_0(M)$ in terms of the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ we see that $S(a\mathbf{u}_1 + b\mathbf{u}_2) = c\mathbf{u}_1 + d\mathbf{u}_2$, where c and d are obtained from the equation:

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} f_{xx}(0, 0) & f_{xy}(0, 0) \\ f_{xy}(0, 0) & f_{yy}(0, 0) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The rank of $\mathbf{S_0}$ can be simply read off as the number of linearly independent rows of the matrix.

(a) In this case $f(x, y) = xy$.

$$f_x(x, y) = y \qquad f_{xx}(x, y) = 0$$
$$f_y(x, y) = x \qquad f_{xy}(x, y) = 1$$
$$f_{yy}(x, y) = 0$$

The matrix of S_0 with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and so

$$S(a\mathbf{u}_1 + b\mathbf{u}_2) = b\mathbf{u}_2 + a\mathbf{u}_1$$

and S_0 has rank 2.

(b) Now $f(x, y) = 2x^2 + y^2$.

$$f_x(x, y) = 4x \qquad f_{xx}(x, y) = 4$$
$$f_y(x, y) = 2y \qquad f_{xy}(x, y) = 0$$
$$f_{yy}(x, y) = 2$$

The matrix of S_0 with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ is

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

and so

$$S(a\mathbf{u}_1 + b\mathbf{u}_2) = 4a\mathbf{u}_1 + 2b\mathbf{u}_2$$

and S_0 has rank 2.

(c) Now $f(x, y) = (x + y)^2$.

$$f_x(x, y) = 2(x + y) \qquad f_{xx}(x, y) = 2$$
$$f_y(x, y) = 2(x + y) \qquad f_{xy}(x, y) = 2$$
$$f_{yy}(x, y) = 2.$$

The matrix of S_0 with respect to the basis $\{u_1, u_2\}$ is

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

and so

$$S(au_1 + bu_2) = 2(a + b)(u_1 + u_2)$$

and S_0 has rank 1.

(d) Now $f(x, y) = xy^2$.

$$f_x(x, y) = y^2 \qquad f_{xx}(x, y) = 0$$
$$f_y(x, y) = 2xy \qquad f_{xy}(x, y) = 2y$$
$$f_{yy}(x, y) = 2x$$

The matrix of S_0 with respect to the basis $\{u_1, u_2\}$ is

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

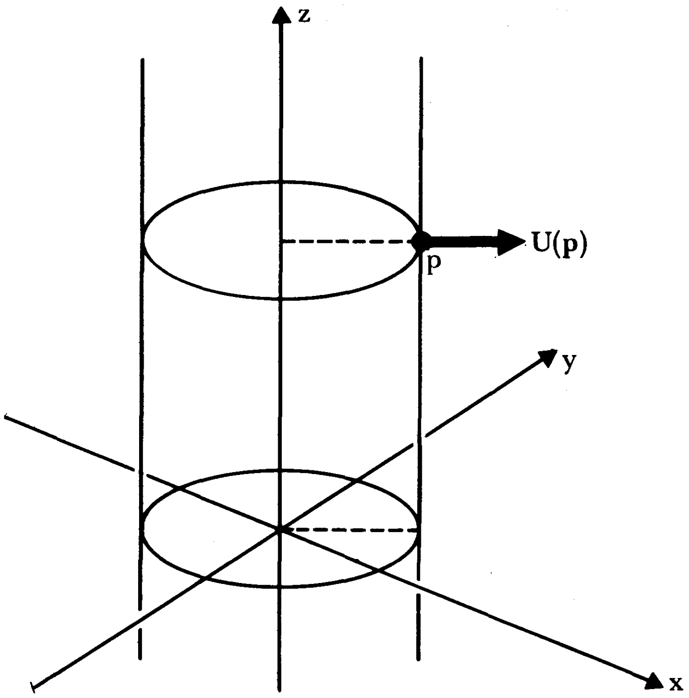
and so

$$S(au_1 + bu_2) = 0$$

and S_0 has rank 0.

6. Page 194, Exercise 4.

(a) The "outwards" unit normal on the cylinder $x^2 + y^2 = r^2$ at the point (p_1, p_2, p_3) is $(p_1, p_2, 0)/r$.



Thus G(M) is the set of all points $(p_1, p_2, 0)/r$ with $p_1^2 + p_2^2 = r^2$; that is, G(M) is the unit circle in the xy plane.

(b) The cone M: $g = (x^2+y^2)^{\frac{1}{2}} - z = 0$ is a "single" cone with positive z coordinates. We must assume that the vertex 0 is omitted, or else M would not be a surface. As a normal vector field on M we can take

$$\begin{aligned}\nabla g &= \frac{x}{(x^2+y^2)^{\frac{1}{2}}}U_1 + \frac{y}{(x^2+y^2)^{\frac{1}{2}}}U_2 - U_3\\ &= \frac{x}{z}U_1 + \frac{y}{z}U_2 - U_3.\end{aligned}$$

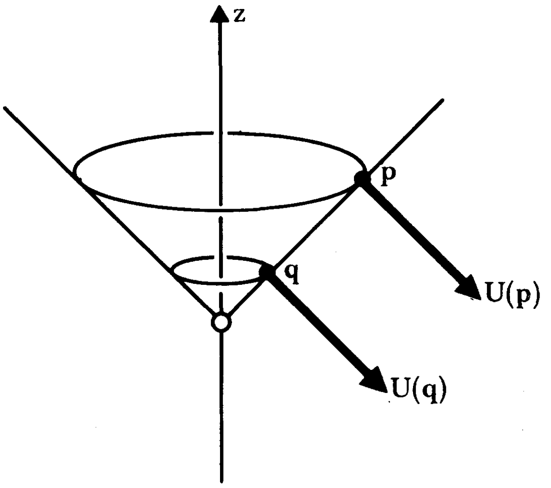
Then

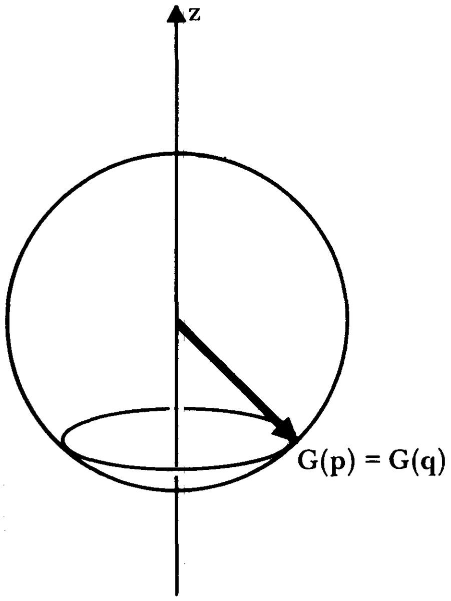
$$\begin{aligned}\|\nabla g\| &= \left(\frac{x^2+y^2}{z^2}+1\right)^{\frac{1}{2}}\\ &= \sqrt{2}\end{aligned}$$

because $x^2 + y^2 = z^2$, and so a unit normal vector field on M is

$$\frac{1}{\sqrt{2}z}(xU_1 + yU_2 - zU_3).$$

Thus G(M) is the set of all points $(p_1, p_2, -p_3)/\sqrt{2}p_3$ with $p_3 = (p_1{}^2 + p_2{}^2)^{\frac{1}{2}}$, which is the circle which is formed by the intersection of the unit sphere Σ with the plane $z = -1/\sqrt{2}$.





(Of course, we could have chosen the opposite unit normal, in which case G(M) would have been the intersection of Σ with the plane $z = 1/\sqrt{2}$.)

(c) As noted in Example V.1.3(2), the unit normal vector field to a plane is parallel; that is, its Euclidean coordinate functions are all constants. Hence if M is a plane then G(M) is a single point. In this case M : g = 0, where g = x + y + z:

$$\nabla g = U_1 + U_2 + U_3$$

and so G(M) is the single point $(1, 1, 1)/\sqrt{3}$.

(d) Using the gradient, we find that

$$(x - 1)U_1 + yU_2 + (z + 2)U_3$$

is a unit normal vector field on M: $(x - 1)^2 + y^2 + (z + 2)^2 = 1$.

Thus

$$\begin{aligned} G(M) &= \{(p_1 - 1, p_2, p_3 + 2): (p_1 - 1)^2 + p_2^2 + (p_3 + 2)^2 = 1\} \\ &= \{(q_1, q_2, q_3): q_1^2 + q_2^2 + q_3^2 = 1\} \\ &= \Sigma. \end{aligned}$$

In other words, in this case G(M) is the whole of the unit sphere.

7. Page 195, Exercise 5.

We use the parametrization

$$x(u, v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)$$

of the torus T. We know that $x_u \times x_v$ is a normal vector field on T. Now

$$x_u(u, v) = (-r\sin u\ \cos v,\ -r\sin u\ \sin v,\ r\cos u)$$

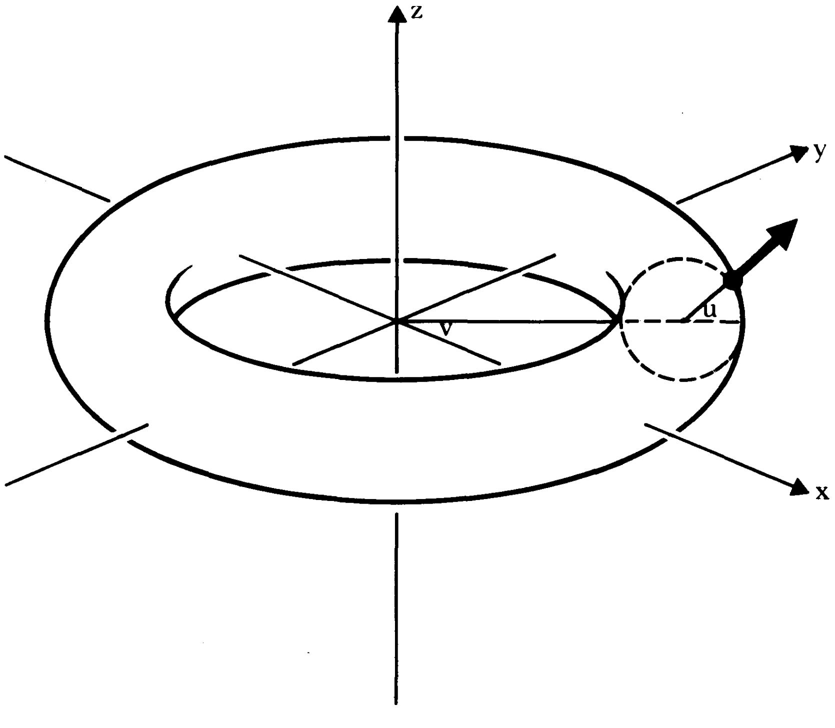
$$x_v(u, v) = (-(R + r\cos u)\sin v,\ (R + r\cos u)\cos v,\ 0)$$

and so

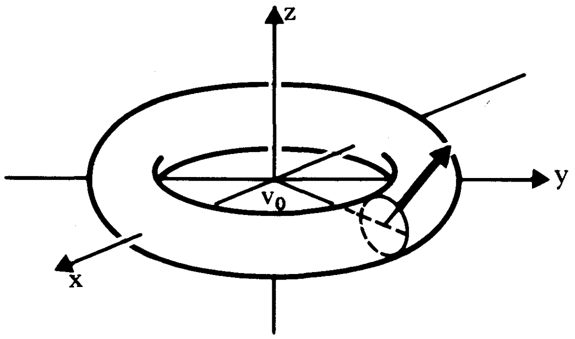
$$(x_u \times x_v)(u, v) = -r(R + r\cos u)(\cos u\ \cos v,\ \cos u\ \sin v,\ \sin u).$$

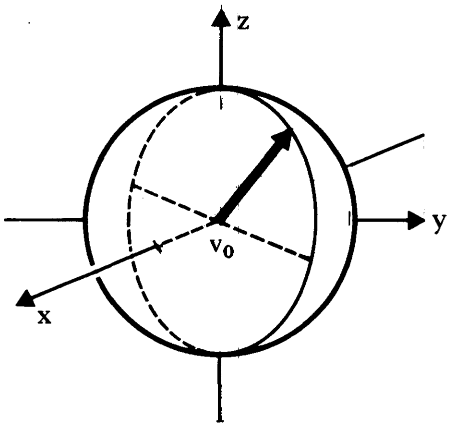
Thus the outward unit normal at x(u, v) is given by

$$(\cos u\ \cos v,\ \cos u\ \sin v,\ \sin u).$$

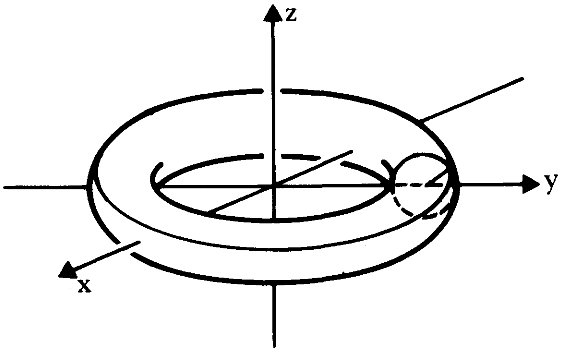


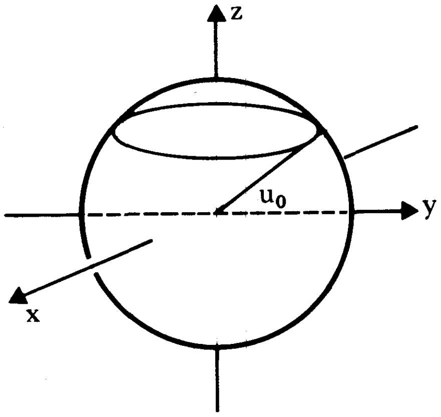
The coordinates of this are identical to those of the geographical parametrization of the unit sphere, except that u and v are interchanged and may each take any real value. A typical meridian of T is the u-parameter curve $v = v_0$: G maps this to the vertical great circle on Σ which is the union of the meridian $u = v_0$ with the meridian $u = v_0 + \pi$ and the North and South poles.

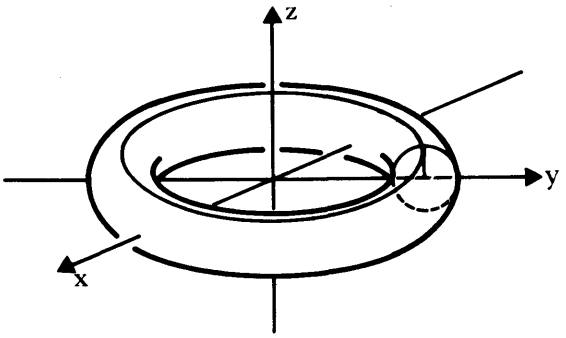


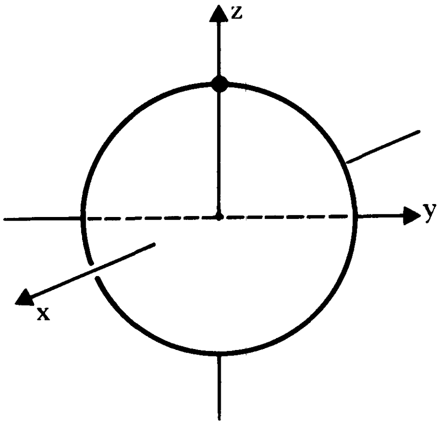


A typical parallel of T is the v-parameter curve $u = u_0$: G maps this to the parallel on Σ given by $v = u_0$ when u_0 is not of the form $n\pi + \pi/2$. These particular values of u_0 correspond to the top circle on T, which is mapped to the North pole on Σ, and the bottom circle on T, which is mapped to the South pole on Σ.

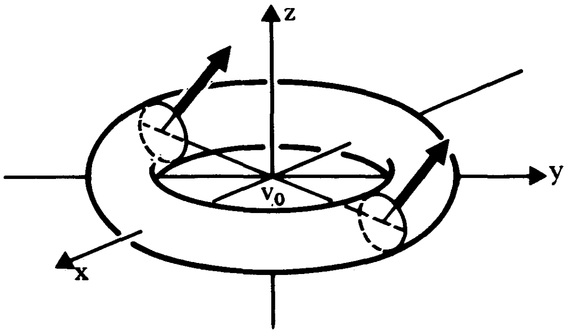


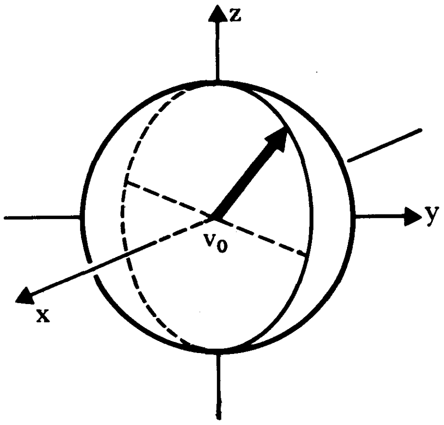






The meridian $v = v_0$ on T has an "opposite" meridian $v = v_0 + \pi$ (the meridian $v = v_0 + 2\pi$ is of course the same as the meridian $v = v_0$). G maps each of these meridians to the whole of the same vertical great circle on Σ.





We have already seen that the North and South poles of Σ are each the image of a whole circle on T. Every other point of Σ lies on just one vertical great circle and so is the image of exactly two points of T, which lie on opposite meridians of T.

8. Page 195, Exercise 6.

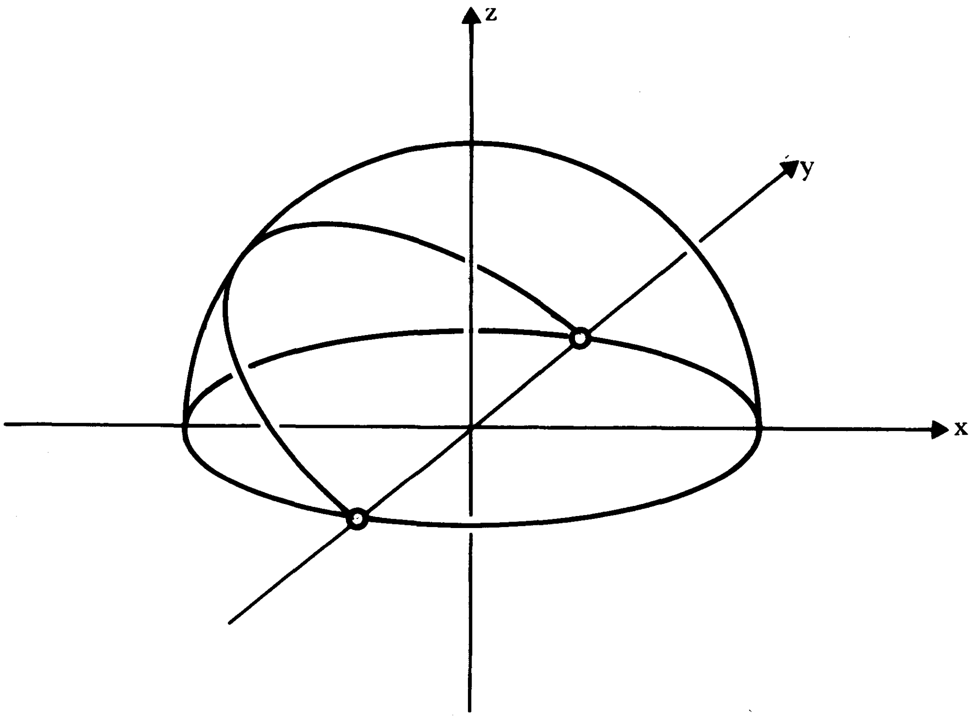
The saddle surface is $M : z = f(x, y)$, where $f(x, y) = xy$. The unit normal vector field U found in Exercise V.1.2 is

$$\frac{-f_xU_1 - f_yU_2 + U_3}{(1 + f_x{}^2 + f_y{}^2)^{\frac{1}{2}}}$$
$$= \frac{-yU_1 - xU_2 + U_3}{(1 + y^2 + x^2)^{\frac{1}{2}}}.$$

The line $y = c$ on M is mapped by G to the set of points

$$\left\{\frac{(-c, -p_1, 1)}{(1 + c^2 + p_1{}^2)^{\frac{1}{2}}} : p_1 \in R\right\}.$$

The point q in Σ belongs to this set if and only if q_3 is positive and $q_1 = -cq_3$. Thus G maps the line $y = c$ to the semicircle which is the intersection of Σ, the plane $x = -cz$ and the set $\{p \in E^3 : z(p) > 0\}$.



Each point of M lies on one of the lines $y = c$ for some real value of c. As c varies over R the images of these lines cover the whole of the northern hemisphere, $\{p \in \Sigma: z(p) > 0\}$.

9. Page 195, Exercise 7.

Let v be a vector tangent to M. Then there is a curve α in M with $\alpha'(0) = v$. Then

$$\begin{aligned} S(v) &= S(\alpha'(0)) \\ &= -(U(\alpha))'(0), \qquad \text{by Exercise V.1.1.} \end{aligned}$$

Now

$$\begin{aligned} G_*(v) &= G_*(\alpha'(0)) \\ &= (G(\alpha))'(0), \qquad \text{by Definition IV.5.3.} \end{aligned}$$

Since G and U have the same Euclidean coordinate functions, $U(\alpha)'(0)$ and $G(\alpha)'(0)$ also have the same coordinates; that is, the vectors $-S(v)$ and $G_*(v)$ are parallel.

V.2 NORMAL CURVATURE

Introduction

This section follows on directly from Section V.1. We also need to use the concepts of *eigenvector* and *eigenvalue* (see *Unit M201 5, Determinants and Eigenvalues*) and a theorem from *Unit M201 24, Orthogonal and Symmetric Transformations.*

In the previous section we saw that S_p is a linear transformation of $T_p(M)$ to itself, and accepted (leaving the proof until Section V.4) that S_p is a symmetric linear transformation; that is,

$$S_p(v)\cdot w = v\cdot S_p(w).$$

You have met symmetric linear transformations before, in *Unit M201 24, Orthogonal and Symmetric Transformations,* where the following important theorem was proved (page 37): if S is a symmetric linear transformation of a finite dimensional vector space V to itself then V has an orthonormal basis consisting of eigenvectors of S. Applying this theorem to S_p, we see that

$$T_p(M) \text{ has a basis } e_1, e_2 \text{ where } \|e_1\| = \|e_2\| = 1,\ e_1\cdot e_2 = 0,$$

and

$$S_p(e_1) = k_1e_1, \quad S_p(e_2) = k_2e_2,$$

where k_1, k_2 are the eigenvalues of S_p corresponding to e_1, e_2. The tangent vectors e_1, e_2 are called *principal vectors* at p, and k_1, k_2 are called *principal curvatures* at p. That is, we make the following definitions.

Definition. The principal curvatures at p are the eigenvalues of S_p.

Definition. The principal vectors at p are the unit eigenvectors of S_p.

Definition. A principal direction at p is the one-dimensional subspace of $T_p(M)$ spanned by a principal vector at p.

Thus, if v is tangent to M at p, $S_p(v)$ is a multiple of v if and only if v is an eigenvector of S_p, which happens if and only if v lies in a principal direction at p. Notice that, with this definition, a principal direction is not oriented; if v is an eigenvector of S_p then v and $-v$ both belong to the same principal direction.

With respect to the basis $\{e_1, e_2\}$ the linear transformation S_p has matrix

$$\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

In the particular case when $k_1 = k_2 = k$ this becomes kI, and so S_p is simply multiplication by the scalar k; that is, every vector of $T_p(M)$ is an eigenvector of S_p with eigenvalue k. In this case we call p *umbilic.*

Definition p is umbilic if S_p is scalar multiplication.

If v is a nonzero eigenvector of S_p with eigenvalue k_i then

$$S_p(v)\cdot v = (k_iv)\cdot v = k_i(v\cdot v)$$

and so

$$k_i = \frac{S_p(v)\cdot v}{v\cdot v}.$$

In particular, since $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors, the eigenvalues of S_p are given by

$$k_i = S_p(\mathbf{e}_i)\cdot\mathbf{e}_i.$$

In fact we have the following theorem.

Theorem V.2.A The principal curvatures at p are the minimum and maximum values of $S_p(\mathbf{u})\cdot\mathbf{u}$ for unit tangent vectors $\mathbf{u}$ in $T_p(M)$. Moreover these minimum and maximum values occur only when $\mathbf{u}$ is a principal vector at p. In particular, $S_p(\mathbf{u})\cdot\mathbf{u}$ is constant if and only if p is umbilic.

Proof Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be an orthonormal eigenvector basis of $T_p(M)$. If $\mathbf{u}$ is any unit tangent vector in $T_p(M)$, we can write

$$\mathbf{u} = \cos\vartheta\ \mathbf{e}_1 + \sin\vartheta\ \mathbf{e}_2$$

for some angle ϑ. Since $S_p(\mathbf{u})$ is a linear transformation

$$\begin{aligned} S_p(\mathbf{u}) &= \cos\vartheta\ S_p(\mathbf{e}_1) + \sin\vartheta\ S_p(\mathbf{e}_2) \\ &= k_1\cos\vartheta\,\mathbf{e}_1 + k_2\sin\vartheta\,\mathbf{e}_2 \end{aligned}$$

and so

$$\begin{aligned} S_p(\mathbf{u})\cdot\mathbf{u} &= (k_1\cos\vartheta\,\mathbf{e}_1 + k_2\sin\vartheta\,\mathbf{e}_2)\cdot(\cos\vartheta\,\mathbf{e}_1 + \sin\vartheta\,\mathbf{e}_2) \\ &= k_1\cos^2\vartheta + k_2\sin^2\vartheta \\ &= k_2 + (k_1 - k_2)\cos^2\vartheta. \end{aligned}$$

As $\cos^2\vartheta$ varies between 1 and 0, $S_p(\mathbf{u})\cdot\mathbf{u}$ varies between k_1 and k_2, which are therefore the minimum and maximum values (not necessarily in that order) of $S_p(\mathbf{u})\cdot\mathbf{u}$ for unit vectors $\mathbf{u}$ in $T_p(M)$. Moreover if $k_1 \neq k_2$, then these extreme values occur only when $\cos^2\vartheta = 0$ or 1, that is when $\mathbf{u} = \pm\mathbf{e}_1$ or $\pm\mathbf{e}_2$.

In the case when p is umbilic, with $k_1 = k_2 = k$, $S_p(\mathbf{u}) = k\mathbf{u}$ for all unit vectors $\mathbf{u}$ in $T_p(M)$, and so $S_p(\mathbf{u})\cdot\mathbf{u} = k$ is constant. Conversely, if $S_p(\mathbf{u})\cdot\mathbf{u}$ is constant then $k_1 = k_2$ and so p is umbilic.

O'Neill approaches in a more geometric way the concepts that we have just defined algebraically. The geometry also naturally leads us to consider the importance of the principal vectors and curvatures, and the *normal curvature function*

$$\mathbf{u} \longmapsto S_p(\mathbf{u})\cdot\mathbf{u} \qquad \mathbf{u} \in T_p(M),\ \|\mathbf{u}\| = 1.$$

We have kept the notation S_p in this introduction to emphasize that S_p is a linear transformation of $T_p(M)$. However, in general it is quite safe to abbreviate S_p to S, as O'Neill has already started doing, for if v is a tangent vector tangent to M at p then S(v) must mean $S_p(v)$.

READ: Section V.2 (pages 195-202), omitting the proof of Theorem 2.5.

Comments

(i) *Page 196: normal curvature* The normal curvature of M in the direction of a unit vector $\mathbf{u}$ tangent to M is defined to be

$$k(\mathbf{u}) = S(\mathbf{u})\cdot\mathbf{u}.$$

Note that, unlike the curvature of a curve, the normal curvature of a surface in a particular direction may be negative.

(ii) *Page 201: $\overline{T}_p(M)$* $\overline{T}_p(M)$ was defined in Exercise IV.3.9 on page 150 to be the plane in E^3 consisting of all points of the form p + v for v in $T_p(M)$. It is the plane through p orthogonal to U(p), and gives a concrete realization of $T_p(M)$.

(iii) *Pages 197 and 201: normal sections and quadratic approximations* O'Neill makes several assumptions in these sections which he does not justify. For the sake of completeness we here point out which results he is assuming and indicate how they may be proved.

(a) Firstly, O'Neill assumes that the position of M in E^3 is of no importance, in other words that a Euclidean isometry F (as defined in Chapter III) maps principal vectors of M to principal vectors of F(M) and makes no difference to the normal and principal curvatures. Since all of these concepts are defined in terms of the shape operator, it is enough to show that F preserves the shape operator; that is,

$$S(F_*(v)) = F_*(S(v))$$

for all vectors v tangent to M. This is in fact true: the proof is given in Section VI.8, and there is no need to read it until you reach that section. Because of this result, we can always simplify our investigation of a neighbourhood of a point p on a surface M by

(i) applying a translation to move p to the origin;

(ii) applying a rotation so that the tangent plane $\overline{T}_p(M)$ becomes the xy plane and any *one* unit vector in $\overline{T}_p(M)$ becomes (1, 0, 0).

(b) Secondly, O'Neill assumes that, if M is transformed so that p becomes the origin and $\overline{T}_p(M)$ becomes the xy plane, then there is a neighbourhood of 0 in which M can be expressed as M:z = f(x, y) for some (differentiable) function f. We can in fact prove this by showing that the projection $\pi : M \longrightarrow E^2$ given by

$$\pi(p_1, p_2, p_3) = (p_1, p_2)$$

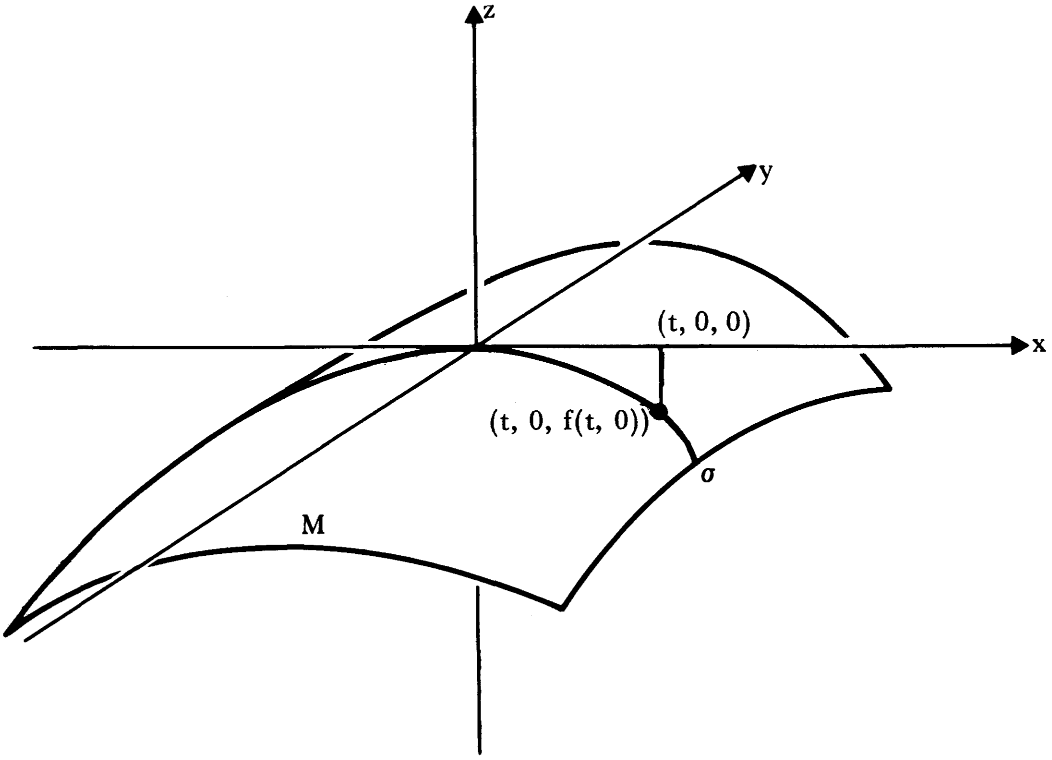
is a diffeomorphism in a neighbourhood of 0 and so has an inverse of the form

$$\pi^{-1}(p_1, p_2) = (p_1, p_2, f(p_1, p_2)).$$

The details of this proof are a little tedious, and you do not need to know them, so we omit them.

(c) Thirdly, O'Neill assumes that the normal section σ, defined on page 197, is genuinely a curve in M, and moreover has a unit-speed reparametrization. Specifically, p is a point of M, u a unit tangent vector in $T_p(M)$, and P is the plane determined by u and U(p); that is, P is the plane through p consisting of all points p + v where v is any linear combination of u and U(p). The normal section σ is defined to be $M \cap P$, and we wish to show that σ is the route of a regular curve in M.

Now, by result (a) we may assume that p is 0, $\overline{T}_p(M)$ is the xy plane (and so $U(p) = \pm (0, 0, 1)$) and u is (1, 0, 0). Then P becomes the plane y = 0. By result (b) there is a neighbourhood of 0 on which M is given by M:z = f(x, y). Hence σ is (t, 0, f(t, 0)) on an interval I of the x-axis.



The function

$$t \longmapsto (t, 0, f(t, 0))$$

is differentiable, so σ is a curve in M. Moreover, its velocity is

$$t \longmapsto (1, 0, f_x(t, 0)),$$

which is never zero: thus σ is regular and so may be given a unit-speed reparametrization.

(iv) *Page 200: Proof of Theorem 2.5* With the definitions given in the Introduction in the Text, this theorem becomes trivial, and so it suffices to prove that the Text definitions of principal curvature, principal vector, principal direction and umbilic are equivalent to those given by O'Neill. This has been done in Theorem V.2.A, so you may omit the proof given here, which makes no appeal to linear algebra.

The formula obtained in Corollary 2.6 is found during the proof of both Theorem 2.5 and Theorem V.2.A.

(v) *Page 202: quadratic approximation of f near (0, 0)* For general f, this approximation is

$$f(x, y) \sim f(0, 0) + f_x(0, 0)x + f_y(0, 0)y +$$

$$\tfrac{1}{2}(f_{xx}(0, 0)x^2 + 2f_{xy}(0, 0)xy + f_{yy}(0, 0)y^2).$$

This is the two-dimensional version of Taylor's approximation, and may be found in *Unit M201 14, Bilinear and Quadratic Forms,* page 25. We shall always use the quadratic approximation $\frac{1}{2}(k_1x^2 + k_2y^2)$, as obtained by shifting the origin and rotating the axes.

Supplementary Comments

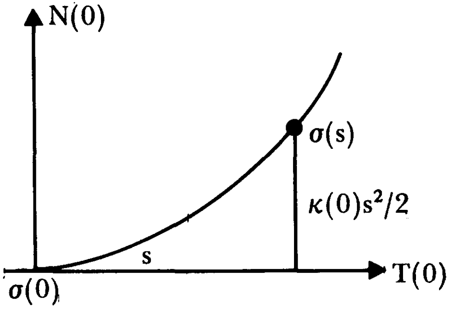
(i) *Page 197: lines -10 to -8* These lines are illustrated by Fig. 5.11 on page 198. The curve σ lies in the plane P and so the tangent vectors $\sigma'(0)$ and

$\sigma''(0)$ both lie in P. However, $\sigma'(0) = T(0)$ and $\sigma''(0) = \kappa(0)N(0)$ and so the tangent vectors T(0) and N(0) both lie in P. Moreover, σ is a curve in M and so $\sigma'(0)$ lies in $T_p(M)$. The only unit tangent vectors at p which lie in both P and $T_p(M)$ are $\pm$u and so $\sigma'(0) = \pm u$. In fact we may choose σ so that $\sigma'(0) = u$. Now the tangent vector N(0) is a unit vector which lies in the plane P and which is orthogonal to the tangent vector $T(0) = \sigma'(0) = u$: the only such vectors are $\pm U(p)$ and so $N(0) = \pm U(p)$.

(ii) *Page 198: shape of σ* The Frenet approximation of a unit speed curve (page 61) tells us that, near 0,

$$\sigma(s) \sim \sigma(0) + s\,T(0) + \kappa(0)\frac{s^2}{2}N(0) + \kappa(0)\tau(0)\frac{s^3}{6}B(0)$$
$$= \sigma(0) + s\,T(0) + \kappa(0)\frac{s^2}{2}N(0)$$

because $\tau(0) = 0$ as σ is a plane curve (lying in the plane P).



$\kappa(0)$ is non-negative by definition, and so σ bends towards N(0) on either side of $\sigma(0)$.

(iii) *Page 198: saddle surface* Since this is given by $z = xy$, along the line $y = x$ we have $z = x^2$ and so the normal section is a parabola bending upwards; along the line $y = -x$ we have $z = -x^2$ and so the normal section is a parabola bending downwards.

(iv) *Page 202: line 1* We are assuming (1) that $p \in M$ is the origin, and so $0 = f(0, 0)$; the Monge patch

$$(u, v) \longmapsto (u, v, f(u, v))$$

has partial velocities $(1, 0, f_u(0, 0))$ and $(0, 1, f_v(0, 0))$ at the origin, and by assumption (2) these are orthogonal to (0, 0, 1) and so

$$f_u(0, 0) = f_v(0, 0) = 0.$$

Summary

Notation

$k(u)$	Page 196, Definition 2.2
k_1, k_2	Page 199, Definition 2.3 and Text, page 19
$\underline{e}_1, \underline{e}_2$	Text, page 19
$T_p(M)$	Page 201, line -4
$\hat{M}$	Page 202, line 15

Definitions

(i)	Normal curvature $(k(u))$ in the u direction	Page 196, Definition 2.2
(ii)	Normal section σ	Page 197, line -11
(iii)	Principal curvatures k_1, k_2	Text, page 19
(iv)	Principal directions	Text, page 19
(v)	Principal vectors e_1, e_2	Text, page 19
(vi)	Umbilic point	Text, page 19
(vii)	Tangent plane $\overline{T}_p(M)$	Page 201, line -4
(viii)	Quadratic approximation $(\hat{M})$ of M near p	Page 202, line 15

Results

(i)	For a curve α in M, $\alpha''\cdot U = S(\alpha')\cdot\alpha'$.	Page 196, Lemma 2.1
(ii)	For a unit-speed curve α in M with $\alpha'(0) = u_p$: $k(u) = \kappa(0)N(0)\cdot U(p)$.	Page 197, line 8
(iii)	For a unit-speed normal section σ, with $\sigma'(0) = u_p$: $k(u) = \pm\kappa_\sigma(0)$.	Page 197, line -7
(iv)	For a unit-speed normal section σ with $\sigma'(0) = u_p$: $k(u) > 0$ implies σ is bending towards U(p) at p, $k(u) < 0$ implies σ is bending away from U(p) at p.	Page 198, first two paragraphs
(v)	The principal curvatures at p are the minimum and maximum values of the normal curvature at p.	Text, page 20, Theorem V.2.A
(vi)	The normal curvature at p is constant if and only if p is umbilic.	Text, page 20 Theorem V.2.A
(vii)	If p is umbilic, all directions at p are principal directions.	Page 200, Theorem 2.5.(1)
(viii)	If p is not umbilic, there are exactly two principal directions at p and these are orthogonal.	Page 200, Theorem 2.5.(2)
(ix)	If e_1, e_2 are principal vectors at p with corresponding principal curvatures k_1 and k_2, and $u = \cos\vartheta\, e_1 + \sin\vartheta\, e_2$, then $k(u) = k_1\cos^2\vartheta + k_2\sin^2\vartheta$.	Page 201, Corollary 2.6
(x)	If k_1, k_2 are the principal curvatures at p, the shape of M near p is approximately the same as the shape of $\hat{M} : z = \frac{1}{2}(k_1x^2 + k_2y^2)$ near **0**.	Page 202, line 15

Techniques

(i)	Finding the principal curvatures and vectors in simple cases.	Text, page 20
(ii)	Finding the quadratic approximation to a surface near a point.	Page 202, line 15

Exercises

Technique (i)

1. Page 202, Exercise 1.

Technique (ii)

2. Page 203, Exercise 4. Describe the appearance of each surface near the origin, using a sketch if possible.

Solutions

1. Page 202, Exercise 1.

(a) At any point p of the cylinder let e_1 be a unit tangent vector along a ruling of the cylinder, e_2 a unit tangent vector tangent to the cross-sectional circle through p, as in Example V.1.3.(3) on page 192. It was shown on page 192 that

$$S(e_1) = 0 = 0e_1$$

$$S(e_2) = -\frac{1}{r}e_2:$$

thus e_1 and e_2 are eigenvectors of S with eigenvalues 0 and $-1/r$. In other words, e_1 and e_2 are principal vectors at p and 0 and $-1/r$ are the principal curvatures at p.

(b) As on page 192, we let $u_1 = U_1(0)$, $u_2 = U_2(0)$. It was shown in the solution to Exercise V.1.3 that

$$S(au_1 + bu_2) = bu_1 + au_2.$$

Thus $au_1 + bu_2$ is a principal vector at p if and only if $bu_1 + au_2$ is a multiple of $au_1 + bu_2$, which happens if and only if either $a = b$ (in which case the corresponding principal curvature is 1) or $a = -b$ (in which case the corresponding principal curvature is -1).

2. Page 203, Exercise 4.

In each case M is the surface $M: z = f(x, y)$ for some function f. We can show that $f(0, 0) = f_x(0, 0) = f_y(0, 0) = f_{xy}(0, 0) = 0$, so that the working on page 202 tells us that $k_1 = f_{xx}(0, 0)$, $k_2 = f_{yy}(0, 0)$ and the quadratic approximation is

$$\hat{M} : z = \tfrac{1}{2}(f_{xx}(0, 0)x^2 + f_{yy}(0, 0)y^2).$$

(a) In this case $f(x, y) = \exp(x^2 + y^2) - 1$.

$$f_x(x, y) = 2x\exp(x^2 + y^2) \qquad f_{xx}(x, y) = (2 + 4x^2)\exp(x^2 + y^2)$$
$$f_y(x, y) = 2y\exp(x^2 + y^2) \qquad f_{xy}(x, y) = 4xy\exp(x^2 + y^2)$$
$$f_{yy}(x, y) = (2 + 4y^2)\exp(x^2 + y^2)$$

The quadratic approximation is the surface

$$\widehat{M} : z = \tfrac{1}{2}(2x^2 + 2y^2) = x^2 + y^2,$$

which is a paraboloid bending upwards with its "south pole" at the origin, so M is bowl-shaped near **0**. For a sketch, invert Fig. 5.18 on page 204.

(b) Now $f(x, y) = \log\cos x - \log\cos y$.

$$f_x(x, y) = -\tan x \qquad f_{xx}(x, y) = -\sec^2 x$$
$$f_y(x, y) = \tan y \qquad f_{xy}(x, y) = 0$$
$$f_{yy}(x, y) = \sec^2 y$$

The quadratic approximation is the surface

$$\widehat{M} : z = \tfrac{1}{2}(-x^2 + y^2) = \tfrac{1}{2}(y^2 - x^2),$$

which is a saddle surface (it cuts the xz plane in the parabola $z = -\frac{1}{2}x^2$, which bends downwards, and the yz plane in the parabola $z = \frac{1}{2}y^2$, which bends upwards) so M is saddle-shaped near **0**. See Fig. 5.19 on page 204.

(c) Now $f(x, y) = (x + 3y)^3$

$$f_x(x, y) = 3(x + 3y)^2 \qquad f_{xx}(x, y) = 6(x + 3y)$$
$$f_y(x, y) = 9(x + 3y)^2 \qquad f_{xy}(x, y) = 18(x + 3y)$$
$$f_{yy}(x, y) = 54(x + 3y)$$

The quadratic approximation is the surface

$$\widehat{M} : z = \tfrac{1}{2}(0x^2 + 0y^2),$$

that is, the xy plane. In fact this tells us nothing about the shape of M near **0** without further information.

V.3 GAUSSIAN CURVATURE

Introduction

This section depends only on Section V.2 and the following two vector identities.

Lemma V.3.A If a, b, c are vectors in E^3 then

$$a \times (b \times c) = (a\cdot c)b - (a\cdot b)c.$$

This occurs in *Unit MST 282 1, Some Basic Tools*, Section 2.3. We use it to prove the next lemma.

Lemma V.3.B (The Lagrange Identity). If v, w, a, b are vectors in E^3 then

$$(v \times w)\cdot(a \times b) = \begin{vmatrix} v\cdot a & v\cdot b \\ w\cdot a & w\cdot b \end{vmatrix}$$

Proof Writing $a \times b = c$ we have

$$\begin{aligned}(v \times w)\cdot(a \times b) &= (v \times w)\cdot c \\ &= v\cdot(w \times c), && \text{by Exercise II.1.4(d),} \\ &= v\cdot(w \times (a \times b)) \\ &= v\cdot((w\cdot b)a - (w\cdot a)b), && \text{by Lemma V.3.A,} \\ &= (v\cdot a)(w\cdot b) - (v\cdot b)(w\cdot a) \\ &= \begin{vmatrix} v\cdot a & v\cdot b \\ w\cdot a & w\cdot b \end{vmatrix}. \end{aligned}$$

Having found the geometric significance of the eigenvectors and eigenvalues of S in the previous section, we now investigate that of the determinant and trace of S. Recall that the determinant of a transformation with matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is ad − bc, and that the trace of this transformation is a + d.

Both of these quantities are independent of the choice of basis.

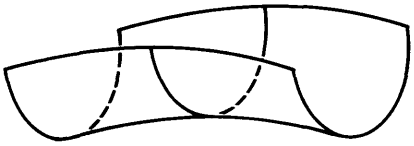
READ: Section V.3 (pages 203-207).

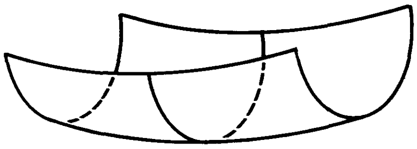
Comments

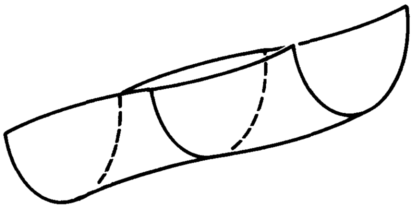
(i) *Page 203: Gaussian and mean curvature* If U is replaced by −U, the Gaussian curvature K is unchanged and so K can be defined unambiguously over the whole of a surface M, even if M is not orientable. The mean curvature H has the usual ambiguity of sign and so cannot be defined over the whole of a nonorientable surface.

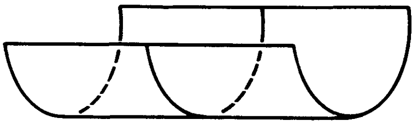
(ii) *Pages 204-205: Remark 3.3* The monkey saddle exemplifies a fact which is implicit in this remark, that if a surface is not of types (1), (2) or (3(a)) (that is, bowl-shaped, saddle-shaped or trough-shaped) near p then p *must* be a planar point; that is, $k_1(p) = k_2(p) = 0$.

The illustration, Fig. 5.20, for the case (3(a)) ($k_1(p) \neq 0$, $k_2(p) = 0$) is a little misleading, as we have no information about how M is curving in the e_2 direction. Each of the diagrams below shows a possibility.









The difficulty is akin to that of distinguishing between the types of stationary point t of a function $g: R \longrightarrow R$ when $g'(t) = g''(t) = 0$.

Supplementary Comments

(i) *Page 206: proof of second half of Lemma 3.4*

$$\begin{aligned} S(v) \times w + v \times S(w) &= (av + bw) \times w + v \times (cv + dw) \\ &= av \times w + dv \times w \\ &= 2H(p)\, v \times w. \end{aligned}$$

(ii) *Page 206: proof of formulas in centre of page* Dotting each side of

$$S(V) \times S(W) = K\, V \times W$$

with $V \times W$ gives

$$(S(V) \times S(W))\cdot(V \times W) = K(V \times W)\cdot(V \times W),$$

which, by the Lagrange identity, is

$$\begin{vmatrix} S(V)\cdot V & S(V)\cdot W \\ S(W)\cdot V & S(W)\cdot W \end{vmatrix} = K \begin{vmatrix} V\cdot V & V\cdot W \\ W\cdot V & W\cdot W \end{vmatrix}.$$

The other formula is obtained similarly.

O'Neill is writing SV for S(V). He frequently drops the parentheses when the meaning is unambiguous—for example he writes Cv for C(v) in Chapter III and fx for f(x) in Chapter IV.

(iii) *Page 207: singular* Since S has matrix

$$\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

with respect to a suitable basis, S is singular if and only if at least one of k_1, k_2 is zero, that is if and only if $\det S = k_1k_2 = K = 0$.

Summary

Notation

K	Page 203, Definition 3.1
H	Page 203, line 7 of the section

Definitions

(i)	Gaussian curvature K	Page 203, Definition 3.1
(ii)	Mean curvature H	Page 203, line 7 of the section
(iii)	Planar point	Page 205, line 4
(iv)	Flat surface	Page 207, Definition 3.6
(v)	Minimal surface	Page 207, Definition 3.6

Results

(i)	$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$.	Text, page 27
(ii)	$(v \times w) \cdot (a \times b) = \begin{vmatrix} v \cdot a & v \cdot b \\ w \cdot a & w \cdot b \end{vmatrix}$	Text, page 27
(iii)	$K = k_1k_2$; $H = (k_1 + k_2)/2$.	Page 203, Lemma 3.2
(iv)	$K(p) > 0 \Rightarrow$ M is bowl-shaped near p; $K(p) < 0 \Rightarrow$ M is saddle-shaped near p; $K(p) = 0 \Rightarrow$ p is a planar point or M is trough-shaped near p.	Page 204, Remark 3.3
	M has a complicated shape near p $\Rightarrow$ p is a planar point.	Text, page 28
(v)	If v and w span $T_p(M)$, then $S(v) \times S(w) = K(p)\, v \times w$ $S(v) \times w + v \times S(w) = 2H(p)\, v \times w$.	Page 205, Lemma 3.4

(vi) If V, W are tangent vector fields which are linearly independent at each point, then

$$K = \frac{\begin{vmatrix} SV\cdot V & SV\cdot W \\ SW\cdot V & SW\cdot W \end{vmatrix}}{\begin{vmatrix} V\cdot V & V\cdot W \\ W\cdot V & W\cdot W \end{vmatrix}}$$

$$H = \frac{\begin{vmatrix} SV\cdot V & SV\cdot W \\ W\cdot V & W\cdot W \end{vmatrix} + \begin{vmatrix} V\cdot V & V\cdot W \\ SW\cdot V & SW\cdot W \end{vmatrix}}{2\begin{vmatrix} V\cdot V & V\cdot W \\ W\cdot V & W\cdot W \end{vmatrix}}$$

Page 206, line 15

(vii) $k_1, k_2 = H \pm (H^2 - K)^{\frac{1}{2}}$. Page 206, Corollary 3.5

Techniques

(i) Finding K and H in simple cases. Page 203, Definition 3.1 and line 7 of the section

(ii) Interpreting K and H geometrically. Page 204, Remark 3.3

Exercises

Technique (i)

1. Find K(0) and H(0) for each of the surfaces in Exercise 4 on page 203.

Technique (ii)

2. Page 207, Exercise 1.

3. Describe the shape of M near p if exactly one of K(p), H(p) is zero.

Theory Exercise

4. Page 207, Exercise 3. (HINT: Use Corollary V.2.6.)

Solutions

1. Writing each of these surfaces as $M: z = f(x, y)$ in turn, we found in the previous section that

$$f(0, 0) = f_x(0, 0) = f_y(0, 0) = f_{xy}(0, 0) = 0.$$

Thus the result of Exercise V.1.2 on page 194 shows that $\mathbf{u}_1$ and $\mathbf{u}_2$ are principal vectors with corresponding principal curvatures $f_{xx}(0, 0)$ and $f_{yy}(0, 0)$. Thus $K(0) = f_{xx}(0, 0)\, f_{yy}(0, 0)$ and $H(0) = \frac{1}{2}(f_{xx}(0, 0) + f_{yy}(0, 0))$. We have already calculated $f_{xx}(0, 0)$ and $f_{yy}(0, 0)$ in these three cases.

(a) $f(x, y) = \exp(x^2 + y^2) - 1$

$f_{xx}(0, 0) = 2$

$f_{yy}(0, 0) = 2$

$K(\mathbf{0}) = 4; H(\mathbf{0}) = 2.$

(b) $f(x, y) = \log \cos x - \log \cos y$

$f_{xx}(0, 0) = -1$

$f_{yy}(0, 0) = 1$

$K(\mathbf{0}) = -1; H(\mathbf{0}) = 0.$

(c) $f(x, y) = (x + 3y)^3$

$f_{xx}(0, 0) = 0$

$f_{yy}(0, 0) = 0$

$K(\mathbf{0}) = 0; H(\mathbf{0}) = 0.$

2. Page 207, Exercise 1.

If $K(\mathbf{p}) < 0$, $k_1(\mathbf{p})$ and $k_2(\mathbf{p})$ must have opposite signs and so cannot be equal: hence $\mathbf{p}$ is not umbilic. If $K(\mathbf{p}) \leqslant 0$ and $\mathbf{p}$ is umbilic, then $k_1(\mathbf{p}) = k_2(\mathbf{p})$ and $k_1(\mathbf{p})k_2(\mathbf{p}) \leqslant 0$, which forces

$$k_1(\mathbf{p}) = k_2(\mathbf{p}) = 0;$$

that is, $\mathbf{p}$ is planar.

3. If $K(\mathbf{p}) = 0$ then at least one of $k_1(\mathbf{p})$, $k_2(\mathbf{p})$ is zero. Were they both zero, $H(\mathbf{p})$ would also be zero. This is not so, so exactly one of the principal curvatures is zero and M is trough-shaped near $\mathbf{p}$.

If $H(\mathbf{p}) = 0$ then $k_1(\mathbf{p})$ and $k_2(\mathbf{p})$ have opposite signs so $K(\mathbf{p}) \leqslant 0$. $K(\mathbf{p})$ is nonzero, so $K(\mathbf{p}) < 0$, so M is saddle-shaped near $\mathbf{p}$.

4. Page 207, Exercise 3.

(a) If $\mathbf{u}$ is a unit vector in $T_{\mathbf{p}}(M)$ we can write

$$\mathbf{u} = \cos \vartheta\, \mathbf{e}_1 + \sin \vartheta\, \mathbf{e}_2$$

for some ϑ, where $\mathbf{e}_1, \mathbf{e}_2$ are orthogonal principal vectors at $\mathbf{p}$.

If v is a unit vector in $T_{\mathbf{p}}(M)$ orthogonal to $\mathbf{u}$ then

$$\mathbf{v} = \cos \varphi\, \mathbf{e}_1 + \sin \varphi\, \mathbf{e}_2$$

where $\varphi = \vartheta + \frac{\pi}{2}$ or $\varphi = \vartheta - \frac{\pi}{2}$.

By Corollary V.2.6,

$$k(\mathbf{u}) = k_1 \cos^2 \vartheta + k_2 \sin^2 \vartheta$$

$$k(\mathbf{v}) = k_1 \cos^2 \varphi + k_2 \sin^2 \varphi.$$

Now

$$\cos\varphi = \cos(\vartheta + \tfrac{\pi}{2}) \quad \text{or} \quad \cos(\vartheta - \tfrac{\pi}{2})$$

$$= -\sin\vartheta \quad \text{or} \quad \sin\vartheta$$

so

$$\cos^2\varphi = \sin^2\vartheta;$$

while

$$\sin\varphi = \sin(\vartheta + \tfrac{\pi}{2}) \quad \text{or} \quad \sin(\vartheta - \tfrac{\pi}{2})$$

$$= \cos\vartheta \quad \text{or} \quad -\cos\vartheta$$

so

$$\sin^2\varphi = \cos^2\vartheta.$$

Thus

$$k(v) = k_1\sin^2\vartheta + k_2\cos^2\vartheta$$

and so

$$\begin{aligned}\tfrac{1}{2}(k(u) + k(v)) &= \tfrac{1}{2}(k_1(\cos^2\vartheta + \sin^2\vartheta) + k_2(\sin^2\vartheta + \cos^2\vartheta)) \\ &= \tfrac{1}{2}(k_1 + k_2) \\ &= H(p).\end{aligned}$$

(b) If $k(\vartheta)$ denotes $k(\cos\vartheta\, e_1 + \sin\vartheta\, e_2)$,

$$\begin{aligned}\int_0^{2\pi} k(\vartheta)d\vartheta &= \int_0^{2\pi} (k_1\cos^2\vartheta + k_2\sin^2\vartheta)d\vartheta \\ &= \pi k_1 + \pi k_2 \\ &= \pi(k_1 + k_2) \\ &= 2\pi H(p).\end{aligned}$$

V.4 COMPUTATIONAL TECHNIQUES

Introduction

This section follows Section V.3 and completes the block introducing the theory of shape. Applications are dealt with in Sections V.5 and V.6.

READ: Section V.4 (pages 210-219).

Comments

(i) *Partial differentiation* In Section IV.2 we saw how partial differentiation came into the definition of the partial velocities x_u and x_v. Now we have encountered their partial derivatives x_{uu}, x_{vv} and $x_{uv} = x_{vu}$ and need to be quite clear how they arise. For instance, the u-parameter curve $v = v_0$ is given by

$$t \longmapsto x(t, v_0).$$

Its velocity vector field is given by

$$t \longmapsto x_u(t, v_0) = \left(\frac{\partial x_1}{\partial u}(t, v_0), \frac{\partial x_2}{\partial u}(t, v_0), \frac{\partial x_3}{\partial u}(t, v_0)\right)_{x(t, v_0)}.$$

The derivative of this vector field, the acceleration of the u- parameter curve, is

$$t \longmapsto \left(\frac{\partial^2 x_1}{\partial u^2}(t, v_0), \frac{\partial^2 x_2}{\partial u^2}(t, v_0), \frac{\partial^2 x_3}{\partial u^2}(t, v_0)\right)_{x(t, v_0)} = x_{uu}(t, v_0).$$

Hence the function x_{uu} gives the acceleration of each u-parameter curve at each point. That is, $x_{uu}(u_0, v_0)$ is the acceleration of the u-parameter curve $v = v_0$ when $u = u_0$ — that is, at the point $x(u_0, v_0)$.

Similar interpretations can be given for x_{vv} and $x_{uv} = x_{vu}$.

We have also used the partial derivative U_v. Now U is a function defined on the domain of the patch x whose value at (u_0, v_0) is a tangent vector normal to the surface at $x(u_0, v_0)$. Hence U is a function

$$U : (u, v) \longmapsto (f(u, v), g(u, v), h(u, v))_{x(u, v)},$$

where f, g and h are real valued functions on the domain of x. We form the partial derivative U_v by taking the partial derivative of each coordinate; that is,

$$U_v : (u, v) \longmapsto \left(\frac{\partial f}{\partial v}(u, v), \frac{\partial g}{\partial v}(u, v), \frac{\partial h}{\partial v}(u, v)\right)_{x(u, v)}.$$

It is possible to define this partial derivative in a coordinate independent way. The composite function $U \circ x^{-1}$ gives us a tangent vector field on the image of the patch and hence we can define the directional derivative $\nabla_{x_v}(U \circ x^{-1})$. But

$$\nabla_{x_v}(U \circ x^{-1}) = (x_v[f \circ x^{-1}], x_v[g \circ x^{-1}], x_v[h \circ x^{-1}])_x$$

$$= \left(\frac{\partial}{\partial v}(f \circ x^{-1} \circ x), \frac{\partial}{\partial v}(g \circ x^{-1} \circ x), \frac{\partial}{\partial v}(h \circ x^{-1} \circ x)\right)_x,$$

by Exercise IV.3.4(b),

$$= \left(\frac{\partial f}{\partial v}, \frac{\partial g}{\partial v}, \frac{\partial h}{\partial v}\right)_x = U_v.$$

Hence $U_v = \nabla_{x_v}(U \circ x^{-1})$, which is defined in a coordinate independent way.

When we partially differentiate the dot product of vector valued functions U and V we obtain the usual Leibnizian result

$$\frac{\partial}{\partial v}(U \cdot V) = U_v \cdot V + U \cdot V_v.$$

Now, in Section V.1 we came across the result $S(\alpha') = -U'$, where α was a curve in the surface and U was the restriction of a normal vector field to the curve. On page 212 we want to look at what happens along the v-parameter curve $u = u_0$ given by

$$\alpha : t \longmapsto x(u_0, t).$$

We know that the velocity vector field is given by

$$\alpha' : t \longmapsto x_v(u_0, t)$$

and the restriction of the normal vector field given on page 211 is

$$U : t \longmapsto U(u_0, t).$$

The derivative of this last vector field is hence

$$t \longmapsto U_v(u_0, t)$$

and we have that

$$S(x_v(u_0, t)) = -U_v(u_0, t),$$

for all u_0 and t, and hence

$$S(x_v) = -U_v$$

as required.

Alternatively we could have obtained this result from first principles. A normal vector field on the image of the patch x is given by the composite function $U \circ x^{-1}$. Hence by the defintion of S

$$S(x_v(u_0, v_0)) = -\nabla_{x_v(u_0, v_0)}(U \circ x^{-1}).$$

But $x_v(u_0, v_0)$ is the initial velocity of the curve

$$t \longmapsto x(u_0, v_0 + t)$$

and hence

$$\begin{aligned} S(x_v(u_0, v_0)) &= -\frac{d}{dt}(U \circ x^{-1} \circ x(u_0, v_0 + t))\Big|_{t=0} \\ &= -\frac{d}{dt}(U(u_0, v_0 + t))\Big|_{t=0} \\ &= -U_v(u_0, v_0), \end{aligned}$$

which is the required result.

(ii) *S is symmetric* To prove that S is symmetric we need to prove that $S(v) \cdot w = S(w) \cdot v$ for each pair v, w of tangent vectors to M at $p = x(u_0, v_0)$. But $v = ax_u + bx_v$ and $w = cx_u + dx_v$ for some real numbers a, b, c, d, where as usual x_u and x_v are evaluated at (u_0, v_0). It follows that

$$\begin{aligned} S(v) \cdot w &= S(ax_u + bx_v) \cdot (cx_u + dx_v) \\ &= (aS(x_u) + bS(x_v)) \cdot (cx_u + dx_v) \\ &= ac\, S(x_u) \cdot x_u + (bc + ad)\, S(x_u) \cdot x_v + bd\, S(x_v) \cdot x_v, \end{aligned}$$

since $S(x_u) \cdot x_v = S(x_v) \cdot x_u$. Similarly

$$S(w) \cdot v = ac\, S(x_u) \cdot x_u + (bc + ad) S(x_u) \cdot x_v + bd\, S(x_v) \cdot x_v = S(v) \cdot w.$$

Since there are enough patches to cover all of the surface we have proved that S is symmetric.

(iii) *Lemma 4.4 and Example 4.5* We need to use the triple scalar product. Given (tangent) vectors **a**, **b** and **c** the triple scalar product is $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$. From Section II.1 we know that the triple scalar product is zero if any two of the vectors **a**, **b** and **c** are scalar multiples of each other. This enables us to prove Lemma 4.4 directly.

By Lemma V.3.4,

$$S(V) \times S(W) = KV \times W = KZ.$$

Hence forming the dot product of both sides with Z we obtain

$$Z \cdot \left(\frac{-\nabla_V Z}{\|Z\|} - V\left[\frac{1}{\|Z\|}\right] Z \right) \times \left(\frac{-\nabla_W Z}{\|Z\|} - W\left[\frac{1}{\|Z\|}\right] Z \right) = K\|Z\|^2.$$

Expanding the left hand side as a linear combination of scalar triple products we find only one term in which the three vectors are distinct and hence

$$\frac{Z \cdot (\nabla_V Z \times \nabla_W Z)}{\|Z\|^2} = K\|Z\|^2,$$

from which the result follows. The formula for H follows similarly.

In Section II.1 we saw further that the triple scalar product can be expressed as a determinant. We know that determinants are linear in each row, from which the equalities on page 218, line 3 follow.

Summary

Notation

E, F, G	Page 210, line 11
U	Page 211, line 3
U_V	Text, page 33
x_{uu}, x_{uv}, x_{vv}	Text, page 33
ℓ, m, n	Page 211, lines -9 to -7
W	Page 214, line 6
h	Page 218, line -6

Definitions

(i)	The speed ($\sqrt{E}, \sqrt{G}$) of the parameter curves	Page 210, line 11
(ii)	The unit normal function U	Page 211, line 3
(iii)	The acceleration (x_{uu} and x_{vv}) of the parameter curves	Page 211, lines 16 and 18, and Text, page 33
(iv)	The support function h	Page 218, line -6

Results

(i) $K = \dfrac{\ell n - m^2}{EG - F^2}$, $H = \dfrac{G\ell + En - 2Fm}{2(EG - F^2)}$. — Page 212, Corollary 4.1.

(ii) S is symmetric. — Page 212, line −2 to Page 213, line 2

(iii) $\ell = U \cdot x_{uu}$, $m = U \cdot x_{uv}$, $n = U \cdot x_{vv}$. — Page 213, Lemma 4.2

(iv) If Z is a nonvanishing normal vector field and $V \times W = Z$ then

$$K = \frac{(Z \cdot \nabla_V Z \times \nabla_W Z)}{\|Z\|^4}$$

$$H = -Z \cdot \frac{(\nabla_V Z \times W + V \times \nabla_W Z)}{2\|Z\|^3}.$$

Page 217, Lemma 4.4

Techniques

(i) Calculating E, F, G, W, U, ℓ, m, n, K and H on a patch x. — Pages 213-215, Example 4.3

(ii) Calculating K and H using linearly independent tangent vector fields. — Page 217, Lemma 4.4.

Exercises

Technique (i)

1. Page 219, Exercise 2.
2. Page 219, Exercise 3.
3. Page 219, Exercise 4.

Technique (ii)

4. Page 222, Exercise 20.

Solutions

1. Page 219, Exercise 2.

$$x(u, v) = (u, v, f(u, v))$$

$$x_u = (1, 0, f_u) \qquad E = 1 + f_u^2$$

$$x_v = (0, 1, f_v) \qquad F = f_u f_v$$

$$G = 1 + f_v^2$$

$$W = (EG - F^2)^{\frac{1}{2}} = (1 + f_u^2 + f_v^2)^{\frac{1}{2}}$$

$$U = x_u \times x_v/W = \frac{1}{W}\begin{vmatrix} U_1 & U_2 & U_3 \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = (-f_u, -f_v, 1)/W$$

$$x_{uu} = (0, 0, f_{uu}) \qquad \ell = U\cdot x_{uu} = f_{uu}/W$$

$$x_{uv} = (0, 0, f_{uv}) \qquad m = U\cdot x_{uv} = f_{uv}/W$$

$$x_{vv} = (0, 0, f_{vv}) \qquad n = U\cdot x_{vv} = f_{vv}/W$$

$$K = \frac{\ell n - m^2}{EG - F^2} = \frac{f_{uu}f_{vv} - f_{uv}^2}{W^4}$$

$$H = \frac{G\ell + En - 2Fm}{2(EG - F^2)}$$

$$= \frac{(1 + f_v^2)f_{uu} + (1 + f_u^2)\, f_{vv} - 2f_u f_v f_{uv}}{2W^3}$$

2. Page 219, Exercise 3.

The image is flat if and only if $K = 0$ and minimal if and only if $H = 0$. The conditions follow from Exercise 1.

3. Page 219, Exercise 4.

For this exercise $f(u, v) = \log \cos v - \log \cos u$.

Hence $f_u = -\frac{(-\sin u)}{\cos u} = \tan u$.

Similarly $f_v = \frac{(-\sin v)}{\cos v} = -\tan v$.

Then $f_{uu} = \sec^2 u$, $f_{uv} = 0$, $f_{vv} = -\sec^2 v$.

Since $(1 + f_u^2)f_{vv} + (1 + f_v^2)f_{uu} - 2f_u f_v f_{uv}$

$$= (1 + \tan^2 u)(-\sec^2 v) + (1 + \tan^2 v)\sec^2 u$$

$$= -\sec^2 u \sec^2 v + \sec^2 v \sec^2 u = 0,$$

the surface is minimal.

The Gaussian curvature is given by

$$K = \frac{f_{uu}f_{vv} - f_{uv}^2}{W^4}$$

$$= -\sec^2 u \sec^2 v/W^4, \qquad \text{where}$$

$$W^2 = 1 + f_u^2 + f_v^2 = 1 + \tan^2 u + \tan^2 v.$$

4. Page 222, Exercise 20.

The saddle surface is the image of the Monge patch $x(u, v) = (u, v, uv)$. In Section IV.2 we defined a ruled surface as one that could be described by a parametrization of the form

$$x(u, v) = \beta_1(u) + v\delta_1(u) \quad \text{or} \quad \beta_2(v) + u\delta_2(v).$$

It is easy to see that the saddle surface is a ruled surface as in this case

$$\begin{aligned} x(u, v) &= (u, 0, 0) + v(0, 1, u) \\ &= (0, v, 0) + u(1, 0, v) \end{aligned}$$

and so we can choose

$$\beta_1(u) = (u, 0, 0), \quad \delta_1(u) = (0, 1, u),$$
$$\beta_2(v) = (0, v, 0), \quad \delta_2(v) = (1, 0, v).$$

The rulings give us straight lines lying within the surface whose directions can be used as a tangent vector fields to M. Thus we obtain tangent vector fields, V and W, such that

$$V(x(u, v)) = (0, 1, u) \text{ at } x(u, v) = (u, v, uv)$$
$$W(x(u, v)) = (1, 0, v) \text{ at } x(u, v) = (u, v, uv).$$

Since these are always linearly independent $Z = V \times W$ is a non-vanishing normal vector field.

By definition

$$Z(x(u, v)) = \begin{vmatrix} U_1 & U_2 & U_3 \\ 0 & 1 & u \\ 1 & 0 & v \end{vmatrix} = (v, u, -1).$$

In order to use the formulas of Lemma V.4.4 we need to calculate $\nabla_V Z$ and $\nabla_W Z$. Now by definition $V(x(u, v))$ and $W(x(u, v))$ are the initial velocities of the curves

$$t \longmapsto x(u, v) + t(0, 1, u) = (u, t + v, u(t + v))$$

and

$$t \longmapsto x(u, v) + t(1, 0, v) = (t + u, v, v(t + u)),$$

i.e.

$$t \longmapsto x(u, t + v)$$

and

$$t \longmapsto x(t + u, v).$$

So

$$\begin{aligned} \nabla_{V(x(u, v))} Z &= \frac{d}{dt}(Z(x(u, t + v))\Big|_{t=0} \\ &= \frac{d}{dt}(t + v, u, -1)\Big|_{t=0} = (1, 0, 0) \end{aligned}$$

and similarly

$$\nabla_{W(x(u,\ v))}Z = (0,\ 1,\ 0).$$

Hence by Lemma V.4.4

$$K(x(u,\ v)) = \frac{(v,\ u,\ -1)\cdot(1,\ 0,\ 0) \times (0,\ 1,\ 0)}{\|(v,\ u,\ -1)\|^4}$$

$$= \frac{-1}{(1 + u^2 + v^2)^2}$$

and

$$H(x(u,\ v)) = \frac{-(v,\ u,\ -1)\cdot((1,\ 0,\ 0) \times (1,\ 0,\ v) + (0,\ 1,\ u) \times (0,\ 1,\ 0))}{2\|(v,\ u,\ -1)\|^3}$$

$$= \frac{uv}{(1 + u^2 + v^2)^{\frac{3}{2}}}$$

These are the results obtained in Example V.4.3(2) apart from the sign of H, which is different due to the fact that Z and U point in opposite directions. H does depend on such a choice while K does not.

V.5 SPECIAL CURVES IN A SURFACE

Introduction

This section follows Section V.4.

We have introduced the shape operator, used it to define normal, Gaussian and mean curvature and obtained effective computational techniques for calculating these quantities. In this section we use this apparatus to describe some special types of curves on a surface.

READ: Section V.5 (pages 223-229).

Comments

(i) *The shape operator S and the unit normal U* The fundamental relationship between the shape operator S and the unit normal U is

$$S(\alpha') = -U',$$

where α is a curve in the surface and we are considering the vector field U restricted to α. We also regularly use the relationship

$$S(\alpha')\cdot\alpha' = \alpha''\cdot U.$$

By the definition of a principal direction we have that α is a principal curve if and only if

$$S(\alpha') = k\alpha'$$

for some real valued function k. Hence principal curves are characterized by the equations $S(\alpha') = k\alpha' = -U'$ and in this case k is the corresponding principal curvature.

This leads immediately to Lemma 5.2(1), and Lemma 5.2(2) follows since

$$k = \frac{S(\alpha')\cdot\alpha'}{\alpha'\cdot\alpha'} = \frac{\alpha''\cdot U}{\alpha'\cdot\alpha'}.$$

There are certain trivial cases where principal curves occur. If for some curve α we find that $U' = 0$ (that is, U is parallel) as in parts of Lemma 5.3 and Lemma 5.6, then it follows that

$$0 = -U' = S(\alpha') = 0\alpha'.$$

Hence α is a principal curve and the corresponding principal curvature at each point is zero. Conversely if α is a principal curve with zero principal curvature then the restriction of U to α is parallel.

By definition a curve α is asymptotic if it lies in an asymptotic direction; that is, if T is the unit tangent vector field then $k(T) = S(T)\cdot T = 0$. Since $T = \alpha'/\|\alpha'\|$ and S is linear, an equivalent definition is that

$$S(\alpha')\cdot\alpha' = 0.$$

Using the identity $S(\alpha')\cdot\alpha' = \alpha''\cdot U$ we see that the curve α is asymptotic if and only if

$$0 = S(\alpha')\cdot\alpha' = \alpha''\cdot U.$$

Again there are trivial cases. If $\alpha'' = 0$ (that is, α is a straight line) then α is an asymptotic curve.

We can use this characterization to throw some light on the remarks on page 226 concerning asymptotic directions and the tangent plane. Suppose that the intersection of the tangent plane $\overline{T}_p(M)$ with M contains a curve α such that $\alpha(0) = p$. Suppose **u** is the normal to the plane: then, since α lies in the plane, we find that $\mathbf{u}\cdot\alpha' = \mathbf{u}\cdot\alpha'' = 0$. But since the plane is tangent to M at p we have that $\mathbf{u} = U(p)$. Hence

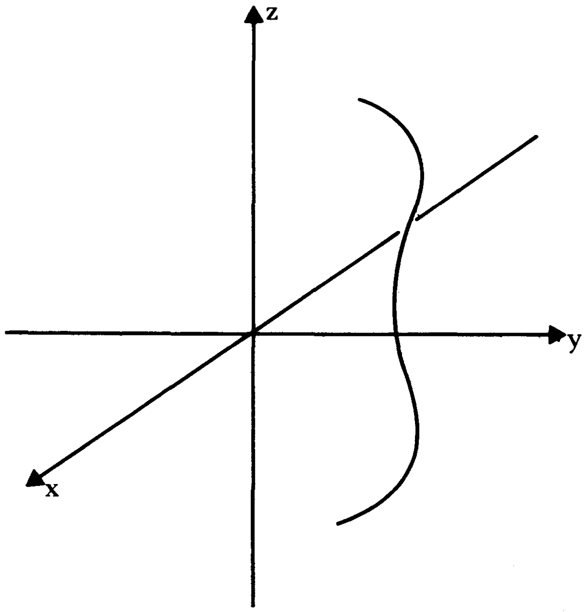
$$S(\alpha'(0))\cdot\alpha'(0) = \alpha''(0)\cdot U(p) = \alpha''(0)\cdot\mathbf{u} = 0$$

and so initially α lies in an asymptotic direction.

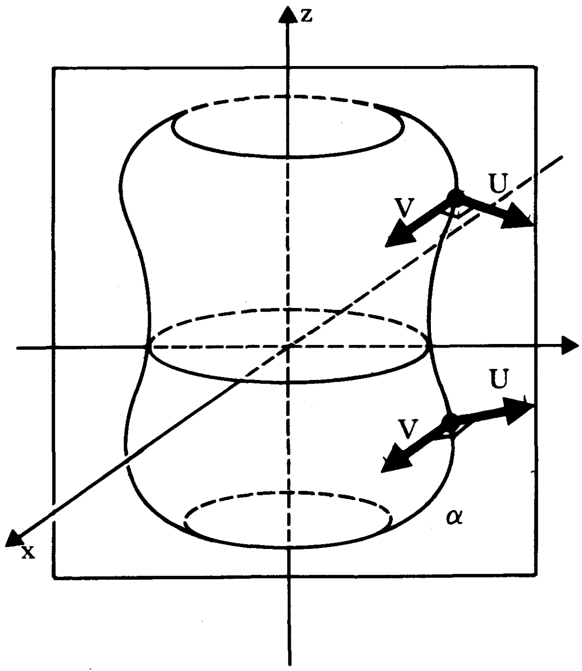
Finally, suppose a curve α is both principal and asymptotic: then $S(\alpha') = k\alpha'$ and $S(\alpha')\cdot\alpha' = 0$. Since for a (regular) curve $\alpha'\cdot\alpha'$ is never zero it follows that k, one of the principal curvatures, is zero.

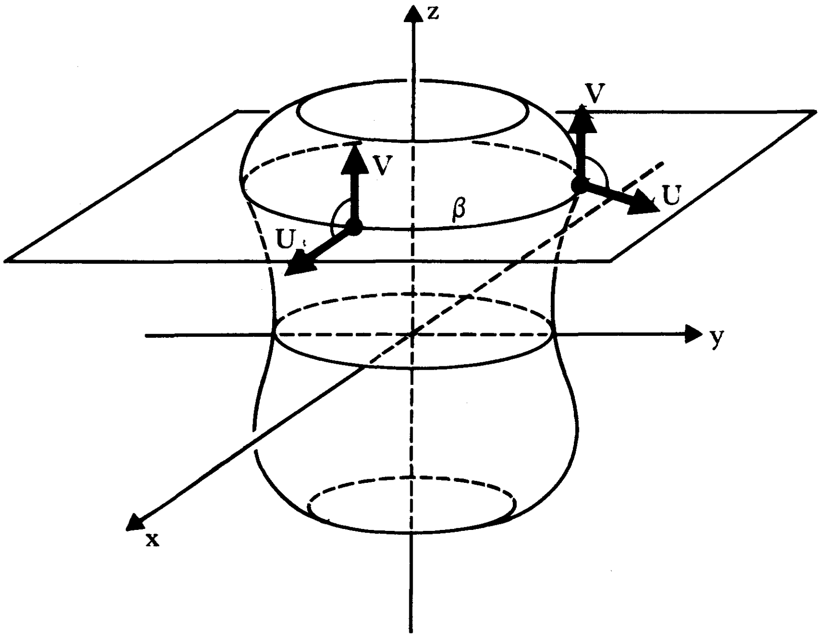
(ii) *Surfaces of Revolution and Lemma 5.3* By Lemma 5.3 we know that if a curve in a surface is the intersection of a plane with the surface and the angle between the normal to the plane and the normal to the surface is constant along the curve then the curve is principal. We can use this result to show that the meridians and parallels on a surface of revolution are principal as in the following example.

Suppose C is a suitable curve in the yz plane.



Revolving this about the z axis we obtain a surface M. The meridians are the intersection of M with vertical planes through the z axis and the parallels are the intersection of M with horizontal planes.





Along each of the curves α and β there is a constant angle between the normal vector fields U and V.

(iii) *Geodesics* In the definition of principal and asymptotic curves we restricted attention to regular curves, while in the definition of geodesics we made no such restriction. The technical reasons for this are beyond the scope of this course. However, since geodesics are constant speed curves any geodesic that is not regular is a zero-speed curve; in other words, it is a constant curve. Conversely for any constant curve α we have $\alpha'' = 0$ and hence α is a geodesic. Consequently we need to restrict attention to regular geodesics. In the examples on pages 228-229 we restrict attention to regular geodesics.

Supplementary Comment

(i) *Page 225: first four lines* If U and V are linearly independent at each point of α then there exists a never zero vector field N on α which is such that N(p) spans the tangent vector space orthogonal to both U(p) and V(p), for each point p on α. Since α' and U' are both orthogonal to U and V it follows that they are both collinear with N and hence with each other.

Summary

Definitions

(i)	Principal curve	Page 223, Definition 5.1
(ii)	Asymptotic direction	Page 225, line 16
(iii)	Asymptotic curve	Page 226, Definition 5.5
(iv)	Geodesic	Page 228, Definition 5.7
(v)	Great circle	Page 229, line 4

Examples

(i)	The geodesics on a plane are the straight lines.	Page 228, Example 5.8(1)
(ii)	The geodesics on a sphere are the great circles.	Page 228, Example 5.8(2)

Results

(i)	A curve α is principal if and only if $S(\alpha') = k\alpha' = -U'$.	Text, page 40 and Page 223, Lemma 5.2(1)
(ii)	If α is principal then the principal curvature is $\alpha''\cdot U/\alpha'\cdot\alpha'$.	Page 223, Lemma 5.2(2)
(iii)	If α is a curve cut from M by a plane P making a constant angle with the surface along α then α is a principal curve.	Page 224, Lemma 5.3
(iv)	The meridians and parallels of a surface of revolution are principal curves.	Page 225, line 9
(v)	(1) If $K(p) > 0$ there are no asymptotic directions at p. (2) If $K(p) < 0$ there are two asymptotic directions at p bisected by the principal directions at an angle ϑ given by $\tan^2\vartheta = -k_1(p)/k_2(p)$. (3) If $K(p) = 0$ and p is a planar point every direction at p is asymptotic and principal; otherwise there is only one asymptotic direction at p, which is also principal.	Page 225, Lemma 5.4.
(vi)	A curve α is asymptotic if and only if $S(\alpha')\cdot\alpha' = -U'\cdot\alpha' = U\cdot\alpha'' = 0$.	Page 226, line -3.

(vii)	The curves given by the intersection of a surface M with the tangent plane $\overline{T}_p(M)$ point in asymptotic directions at p.	Text, page 41
(viii)	A surface is minimal if and only if there exist two orthogonal asymptotic directions at each of its points.	Page 227, line 4
(ix)	A ruled surface has Gaussian curvature $K \leqslant 0$. $K = 0$ if and only if the unit normal U is parallel along each of the rulings.	Page 227, Lemma 5.6
(x)	A straight line in a surface is asymptotic.	Page 227, line -4
(xi)	A geodesic has constant speed.	Page 228, line 18
(xii)	A straight line in a surface is a geodesic.	Page 228, line 21
(xiii)	For unit-speed geodesics, $S(T) = \kappa T - \tau B$.	Page 228, line -11.

Techniques

(i) Using Results (i) – (iv) and (vi) to recognize or find principal curves or asymptotic curves.

(ii) Using the definition to recognize a geodesic.

Exercises

Technique (i)

1. Page 230, Exercise 2. Apply as many as possible of the techniques based on Results (i) – (iv) and (vi) to the three curves.

Technique (ii)

2. Are the curves of Exercise 1 geodesics?

Theory Exercise

3. Page 230, Exercise 7. This exercise gives an approach to curves on a surface that encompasses principal curves, asymptotic curves and geodesics.

Solutions

1. Page 230, Exercise 2.

(a) The top circle α on the torus.

The standard parametrization of the torus is

$$x(u, v) = ((R + r\cos u)\cos v,\ (R + r\cos u)\sin v,\ r\sin u).$$

The top circle is the v-parameter curve $u = \frac{\pi}{2}$. This is given by

$$\alpha : t \longmapsto x(\tfrac{\pi}{2}, t) = (R\cos t,\ R\sin t,\ r).$$

By symmetry arguments the unit vector field normal to the surface along this curve points in the direction (0, 0, 1). Hence on the curve the normal vector field is $U:t\longmapsto(0,0,1)_{\alpha(t)}$.

To apply the techniques we need to compute the following:

$$\alpha' : t\longmapsto(-R\sin t,\ R\cos t,\ 0)$$
$$\alpha'' : t\longmapsto(-R\cos t,\ R\sin t,\ 0)$$
$$U' : t\longmapsto(0,0,0)_{\alpha(t)}$$
$$S(\alpha') = -U' = (0,0,0).$$

Now we can apply Results (i) – (iv), (vi).

(i) Since $U' = \mathbf{0}$ it follows that $S(\alpha') = 0\alpha'$ and hence α is principal.

(ii) The principal curvature of α is 0.

(iii) α is the intersection of the plane $z = r$ with the torus, and the normal to that plane is equal to the normal to the surface along α. Hence again α is a principal curve.

(iv) α is a parallel and hence is a principal curve.

(vi) Since $S(\alpha') = \mathbf{0}$ it follows that $S(\alpha')\cdot\alpha' = 0$ and hence α is asymptotic.

(b) The outer equator β of the torus.

Using the same parametrization as in (a) the outer equator is given by

$$\beta : t\longmapsto x(0,t) = ((R+r)\cos t,\ (R+r)\sin t,\ 0).$$

By symmetry arguments the unit vector field normal to the surface along this curve points radially outwards (or alternatively inwards) and hence is given by

$$U : t\longmapsto(\cos t,\ \sin t,\ 0)_{\beta(t)}.$$

We calculate

$$\beta' : t\longmapsto(-(R+r)\sin t,\ (R+r)\cos t,\ 0)$$
$$\beta'' : t\longmapsto(-(R+r)\cos t,\ -(R+r)\sin t,\ 0)$$

and $U' : t\longmapsto(-\sin t,\ \cos t,\ 0)$.

Now we can apply Results (i) – (iv), (vi).

(i) Since $S(\beta') = -U' = -\dfrac{\beta'}{(R+r)}$ the curve β is principal.

(ii) The principal curvature along β is the constant function $-1/(R+r)$.

(iii) β is the intersection of the torus with part of the plane $z = 0$. The normal to this plane is vertical while the normal to the torus along β is always horizontal. Therefore they make a constant angle of $\frac{\pi}{2}$ with each other and so β is a principal curve.

(iv) β is a parallel and hence a principal curve.

(vi) Since $\beta''\cdot U = -(R+r) \neq 0$ we see that β is not an asymptotic curve.

(c) The x axis on $M: z = xy$.

The x axis is the curve

$$\gamma : t \longmapsto (t, 0, 0).$$

Since M is defined implicitly as $M{:}g = 0$, where $g = z - xy$, we can use ∇g to obtain a unit vector field normal to M. This is

$$U = \frac{(-y, -x, 1)}{(1 + x^2 + y^2)^{\frac{1}{2}}}.$$

Its restriction to γ is

$$U : t \longmapsto \frac{(0, -t, 1)}{(1 + t^2)^{\frac{1}{2}}}.$$

Next we find:

$$\gamma' : t \longmapsto (1, 0, 0)$$
$$\gamma'' : t \longmapsto (0, 0, 0)$$
$$U' : t \longmapsto \frac{(0, -1, -t)}{(1 + t^2)^{\frac{3}{2}}}.$$

Now we can apply Results (i) and (vi).

(i) Since U' and γ' are not collinear, γ is not a principal curve.

(vi) Since $\gamma'' = \mathbf{0}$ it follows that $U\cdot\gamma'' = 0$ and hence γ is an asymptotic curve.

2. Page 230, Exercise 2.

(a) Since $\alpha'' \neq \mathbf{0}$ while $\alpha''\cdot U = 0$ the acceleration is not normal to the surface and hence α is not a geodesic.

(b) Since $\beta'' = -(R + r)U$ the curve β is a geodesic.

(c) Since $\gamma'' = \mathbf{0} = 0U$ the curve γ is a geodesic.

3. Page 230, Exercise 7.

(a) Since T, V, U form a frame field orthonormal expansion gives

$$T' = (T'\cdot T)T + (T'\cdot V)V + (T'\cdot U)U$$
$$V' = (V'\cdot T)T + (V'\cdot V)V + (V'\cdot U)U$$
$$U' = (U'\cdot T)T + (U'\cdot V)V + (U'\cdot U)U.$$

Differentiating the equations

$$T\cdot T = V\cdot V = U\cdot U = 1$$

and

$$T\cdot V = V\cdot U = U\cdot T = 0$$

we obtain

$$T'\cdot T = V'\cdot V = U'\cdot U = 0$$

and

$$T'\cdot V + V'\cdot T = V'\cdot U + U'\cdot V = U'\cdot T + T'\cdot U = 0.$$

Setting $g = T'\cdot V$, $k = T'\cdot U$ and $t = V'\cdot U$ the result follows.

Since $T = \alpha'$ it follows that

$$k = T'\cdot U = \alpha''\cdot U = S(\alpha')\cdot\alpha' = S(T)\cdot T.$$

Since $t = V' \cdot U$ it follows that

$$t = V' \cdot U = -U' \cdot V = S(\alpha') \cdot V = S(T) \cdot V.$$

(b) α is geodesic $\iff \alpha''$ is collinear to U

$\iff T'$ is collinear to U

$\iff g = 0.$

α is asymptotic $\iff S(\alpha') \cdot \alpha' = 0$

$\iff S(T) \cdot T = 0$

$\iff k = 0.$

α is principal $\iff \alpha'$ and $S(\alpha')$ are collinear

$\iff T$ and $-U'$ are collinear

$\iff t = 0.$

V.6 SURFACES OF REVOLUTION

Introduction

This section follows on from Sections IV.2 and V.4.

In Example IV.2.5 we introduced a way of parametrizating surfaces of revolution and in Section V.4 we obtained formulas for the Gaussian and mean curvature of surfaces given by a parametrization. In this section we tie these two sections together and work out explicit formulas for curvature of surfaces of revolution.

We then describe surfaces of revolution that are minimal or have constant Gaussian curvature.

READ: Section V.6 (pages 234-242) omitting cases 1, 2 and 3 on pages 240-241.

Comment

(i) *Page 235: first three lines* Since S maps $T_p(M)$ linearly to itself and $\{x_u, x_v\}$ is a basis for $T_p(M)$ we must have

$$S(x_u) = ax_u + bx_v$$
$$S(x_v) = cx_u + dx_v$$

for some functions a, b, c, d. Forming dot products with x_u and x_v we obtain the equations

$$\ell = aE + bF \qquad m = aF + bG$$
$$m = cE + dF \qquad n = cF + dG.$$

Since $F = m = 0$ it follows that

$$b = c = 0 \text{ and } a = \ell/E,\ d = n/G.$$

Hence $S(x_u) = (\ell/E)x_u$ and $S(x_v) = (n/G)x_v$.

Supplementary Comment

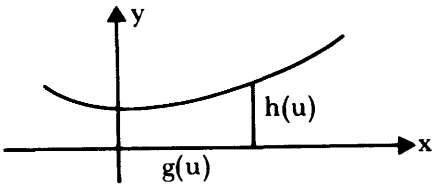
(i) *Page 236: Theorem 6.2*

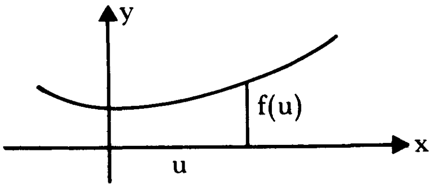
Case 1: If $g = c$ (a constant) then

$$x(u, v) = (c, h(u)\cos v,\ h(u)\sin v)$$

and M belongs to the plane $x = c$. This is a plane parallel to the yz plane and it passes through the point (c, 0, 0).

Case 2: If g' is never zero then either g is strictly increasing or strictly decreasing. In either case the profile curve passes over each point of the x axis once only and so we can reparametrize it so that it is the graph of a function f. That is, we can replace g and h by the functions u and f.





Then by the formulas on page 235

$$2H = k_\pi + k_\mu = \frac{1}{f(1+f'^2)^{\frac{1}{2}}} - \frac{f''}{(1+f'^2)^{\frac{3}{2}}}.$$

Since M is minimal $H = 0$, and hence multiplying through by $f(1+f'^2)^{\frac{3}{2}}$ we obtain

$$1 + f'^2 = ff''.$$

To solve this equation we first differentiate it to obtain

$$2f'f'' = ff''' + f'f''.$$

Hence $f'f'' = ff'''$;

$$\frac{f'}{f} = \frac{f'''}{f''}.$$

Integrating we find

$$\log f = \log f'' - \log c$$

and hence $f'' = cf$ for some constant c.

Since the graph of f does not intersect the axis, we can assume that f is positive and hence so is $f'' = (1 + f'^2)/f$. Hence $c > 0$ and we can write $c = 1/a^2$. Then we can solve

$$f'' = \frac{1}{a^2} f$$

to obtain

$$f(u) = d \cosh\left(\frac{u}{a} + b\right),$$

where d and b are constants. Substituting in the original equation we find that $d = a$ and hence the solution is as given.

Summary

Definitions

(i)	*Standard* parametrization of a surface of revolution: $x(u, v) = (g(u), h(u) \cos v,\ h(u) \sin v)$	Page 234, line 12
(ii)	*Special* parametrization of a surface of revolution passing at most once over each point of the axis: $x(u, v) = (u, h(u) \cos v,\ h(u) \sin v)$	Page 235, line -10
(iii)	*Canonical* parametrization of a surface of revolution: as for the standard parametrization but with a unit speed profile curve, i.e. $g'^2 + h'^2 = 1$	Page 238, line 12

Examples

(i)	Torus of revolution	Page 235, Example 6.1(1)
(ii)	Catenoid	Page 236, Example 6.1(2)
(iii)	Bugle surface	Page 241, Example 6.6

Results

(i) For the standard parametrization

$$k_\mu = -\frac{\begin{vmatrix} g' & h' \\ g'' & h'' \end{vmatrix}}{(g'^2 + h'^2)^{\frac{3}{2}}} \qquad k_\pi = \frac{g'}{h(g'^2 + h'^2)^{\frac{1}{2}}}$$

$$\text{and } K = -\frac{g'\begin{vmatrix} g' & h' \\ g'' & h'' \end{vmatrix}}{h(g'^2 + h'^2)^2}.$$

Page 235, lines 7 and 9

(ii) For the special parametrization

$$k_\mu = \frac{-h''}{(1 + h'^2)^{\frac{3}{2}}}, \qquad k_\pi = \frac{1}{h(1 + h'^2)^{\frac{1}{2}}}$$

$$\text{and} \quad K = \frac{-h''}{h(1 + h'^2)^2}.$$

Page 235, lines -6 to -4

(iii) If a surface of revolution is minimal then it is contained in either a plane or a catenoid.

Page 236, Theorem 6.2

(iv) For the canonical parametrization

$$K = -h''/h.$$

Page 238, Lemma 6.3

(v) A surface of revolution with curvature K is obtained as follows. First find a function h such that $h'' + Kh = 0$, with $h(0) > 0$ and $|h'(0)| < 1$ and then find a function g such that

$$g(u) = \int_0^u (1 - h'(t)^2)^{\frac{1}{2}}\, dt.$$

Then the profile curve for the surface is $\alpha(u) = (g(u), h(u), 0)$.

Page 239, lines -13 to -2

(vi) For a torus of revolution

$$K = \frac{\cos u}{r(R + r\cos u)}.$$

Page 235, Example 6.1(1)

(vii) For a catenoid

$$K = \frac{-1}{c^2 \cosh^4(u/c)}.$$

Page 236, Example 6.1(2)

(viii) A canonical parametrization of the catenoid is given by the unit speed profile curve

$$\beta(s) = (\sinh^{-1} s, (1 + s^2)^{\frac{1}{2}})$$

and for this parametrization $K(s) = \dfrac{-1}{(1 + s^2)^2}$.

Page 238, Example 6.4

(ix) The bugle surface has constant curvature $-1/c^2$.

Techniques

(i)	Calculating the principal and Gaussian curvature for surfaces of revolution using the formulas for standard, special and canonical parametrizations.	Page 235, lines 7 and 9 and Page 235, lines −6 to −4 and Page 238, Lemma 6.3.
(ii)	Finding surfaces of revolution with prescribed Gaussian curvature.	Page 239, lines −13 to −2.

Exercises

Technique (i)

1. Let α be the profile curve that is the top half of the ellipse $x^2/a^2 + y^2/b^2 = 1$. We can parametrize this as follows:

$$\alpha(u) = (a \cos u,\ b \sin u) \qquad \text{where } 0 < u < \pi.$$

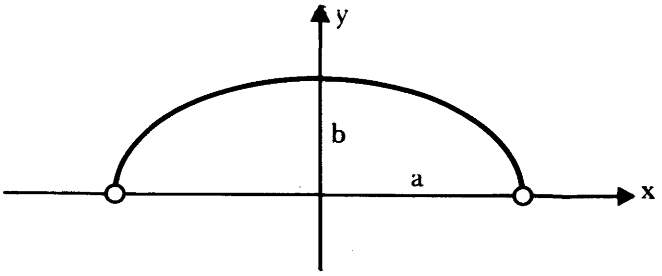
Find a special parametrization and use both this and the given parametrization to calculate the Gaussian curvature of the surface of revolution obtained by revolving about the x axis. This surface is an ellipsoid.

Technique (ii)

2. Find those surfaces of revolution with zero Gaussian curvature. Use the technique described in Result (v) and also consider the case when $|h'(0)| = 1$.

Solutions

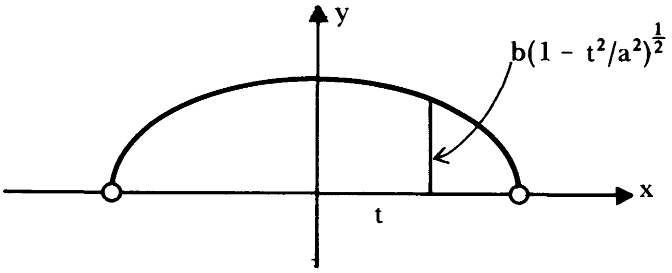
1. The ellipse appears as follows.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

A special parametrization is given by the profile curve

$$\beta(t) = (t,\ b(1 - t^2/a^2)^{\frac{1}{2}}), \qquad \text{where } a > t > -a.$$



We have been given a standard parametrization with $g(u) = a \cos u$ and $h(u) = b \sin u$ and hence using the formula (2) on page 235

$$K = -\frac{g'\begin{vmatrix} g' & h' \\ g'' & h'' \end{vmatrix}}{h(g'^2 + h'^2)^2}$$

so $$K(u) = \frac{a \sin u \begin{vmatrix} -a \sin u & b \cos u \\ -a \cos u & -b \sin u \end{vmatrix}}{b \sin u\,(a^2 \sin^2 u + b^2 \cos^2 u)^2}$$

$$= \frac{a}{b}\,\frac{(ab \sin^2 u + ab \cos^2 u)}{(a^2 \sin^2 u + b^2 \cos^2 u)^2} = \frac{a^2}{(a^2 \sin^2 u + b^2 \cos^2 u)^2}.$$

For the special parametrization

$$h(t) = b(1 - t^2/a^2)^{\frac{1}{2}} = \frac{b}{a}(a^2 - t^2)^{\frac{1}{2}}$$

$$h'(t) = -bt/a(a^2 - t^2)^{\frac{1}{2}}, \quad h''(t) = -ba/(a^2 - t^2)^{\frac{3}{2}}$$

$$1 + h'(t)^2 = \frac{a^4 + (b^2 - a^2)t^2}{a^2(a^2 - t^2)}.$$

Hence $$K(t) = \frac{-h''(t)}{h(t)(1 + h'(t)^2)^2} = \frac{a^6}{(a^4 + (b^2 - a^2)t^2)^2}.$$

If we evaluate the second formula for K at the point $t = a \cos u$ we should obtain the first formula.

$$K(a \cos u) = \frac{a^6}{(a^4 + (b^2 - a^2)a^2 \cos^2 u)^2}$$

$$= \frac{a^2}{(a^2(1 - \cos^2 u) + b^2 \cos^2 u)^2}$$

$$= \frac{a^2}{(a^2 \sin^2 u + b^2 \cos^2 u)^2}, \qquad \text{as we expected.}$$

2. Since $K = 0$ we need to solve

$$h'' + 0h = h'' = 0$$

with $h(0) > 0$ and $|h'(0)| \leqslant 1$.

We obtain $h(t) = at + b$ with $|a| \leqslant 1$. We next find the function

$$g(u) = \int_0^u (1 - h'(t)^2)^{\frac{1}{2}} dt = (1 - a^2)^{\frac{1}{2}} u.$$

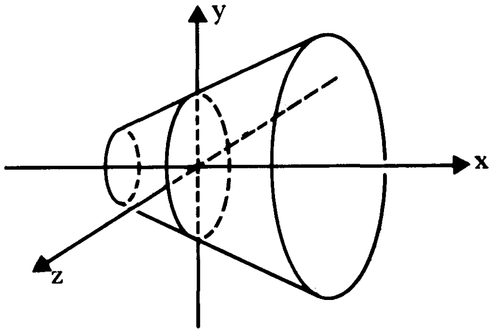
Hence the surface of revolution is obtained by revolving the curve

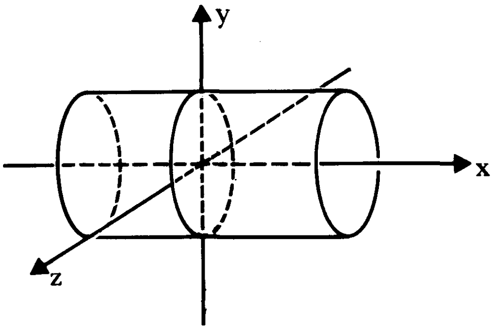
$$t \longmapsto ((1 - a^2)^{\frac{1}{2}} t, at + b)$$

about the x axis. Now, if $1 - a^2 \neq 0$ (that is, $|h'(0)| < 1$) we can reparametrize the profile curve so that $g(t) = t$. Hence we see that the surface can be obtained by revolving the straight line

$$t \longmapsto (t, ct + b).$$

The surfaces we obtain are parts of cylinders (when a = c = 0) and parts of cones.





On the other hand, if $1 - a^2 = 0$ (that is, $|h'(0)| = 1$) then $g' = 0$ and so g is identically zero. Thus the surface is part of the yz plane.

FURTHER EXERCISES AND SOLUTIONS

Section V.1

Recommended Exercise

Page 195, Exercise 9. This continues Exercise II.5.7. You will have to assume that W is also a tangent vector field on M.

Solution

9.
$$
\begin{aligned}
& U\cdot W = 0 \\
\Rightarrow\ & \nabla_V U\cdot W + \nabla_V W\cdot U = 0 \\
\Rightarrow\ & S(V)\cdot W = \nabla_V W\cdot U. \\
& S(V)\cdot W = S(W)\cdot V \\
\Longleftrightarrow\ & \nabla_V W\cdot U - \nabla_W V\cdot U = 0 \\
\Longleftrightarrow\ & [V, W]\cdot U = 0.
\end{aligned}
$$

Section V.2

Technique Exercise

Page 202, Exercise 3.

Solution

$$
\begin{aligned}
p &= \alpha_n(0) = (r, 0, 0) \\
U(p) &= (1, 0, 0) \\
\alpha_n''(t) &= (-r\cos t,\ -r\sin t,\ \pm n(n-1)t^{n-2}) \\
\alpha_n''(0) &= (-r, 0, 0) \quad \text{if} \quad n > 2; \qquad \alpha_2''(0) = (-r, 0, \pm 2) \\
\alpha_n''(0)\cdot U(p) &= -r.
\end{aligned}
$$

Section V.3

Technique Exercises

Pages 207-8, Exercises 2, 5(a), (b). The final zero in 2(b) and 2(c) should be in bold type. (HINT for Exercise 2: Express each condition in terms of the matrix for S_p with respect to $\{\mathbf{u}_1, \mathbf{u}_2\}$.)

Other Recommended Exercise

Page 208, Exercise 4(a). This gives an alternative proof of Corollary V.3.5 by finding the equation whose roots are the principal curvatures.

Solutions

2(a). $\mathbf{u}_1$ and $\mathbf{u}_2$ are principal vectors.

2(b). $(\mathbf{u}_1 \pm \mathbf{u}_2)/\sqrt{2}$ are principal vectors and $K(p) = 0$.

2(c). $K(p) = 0$

2(d). $\mathbf{u}_1$ and $\mathbf{u}_2$ are principal vectors or $H(p) = 0$.

5(a). A single point; two intersecting straight lines; one straight line.

5(b). An ellipse, the empty set; hyperbola, hyperbola; two parallel straight lines, the empty set.

4(a). If T is any 2-dimensional linear operator, with matrix A, then $\det(A - kI) = k^2 - (\text{Trace } A)\, k + \det A$, as may be verified by evaluating each side.

Section V.4

Technique Exercises

Pages 219-221, Exercises 1, 17.

Other Recommended Exercises

Page 220, Exercises 5 and 6. These extend the exercises on curvature of a Monge patch.

Page 220, Exercise 9. This deals with the matrix representation of S in simple cases.

Pages 220-221, Exercises 10, 11. These deal with a coordinate description of principal vectors.

Solutions

1. $\ell = -r\cos^2 v$ $\qquad W = r^2 \cos v$

$m = 0$ $\qquad K = 1/r^2$

$n = -r$

$$Z = \tfrac{1}{2}\nabla g = \frac{x}{a^2}U_1 + \frac{y}{b^2}U_2 - \frac{z}{c^2}U_3$$

$$\nabla_V Z = \frac{v_1}{a^2}U_1 + \frac{v_2}{b^2}U_2 - \frac{v_3}{c^2}U_3$$

$$Z\cdot\nabla_V Z \times \nabla_W Z = \frac{1}{a^2b^2c^2}\begin{vmatrix} x & y & -z \\ v_1 & v_2 & -v_3 \\ w_1 & w_2 & -w_3 \end{vmatrix}$$

$$= \frac{-1}{a^2b^2c^2}\, X\cdot V \times W = -\frac{1}{a^2b^2c^2}\, X\cdot Z$$

$$= \mp\frac{1}{a^2b^2c^2}$$

$$K = \mp\frac{1}{a^2b^2c^2\,\|Z\|^4}$$

$$h = X\cdot U = \frac{X\cdot Z}{\|Z\|} = \pm\frac{1}{\|Z\|}$$

$$K = \mp h^4/a^2b^2c^2$$

5. $K = -36r^2/(1 + 9r^4)^2$

No

6. $K = 4\varepsilon/a^2b^2\left(1 + 4\dfrac{u^2}{a^4} + 4\dfrac{v^2}{b^4}\right)^2$

9(a). $S(x_u) = ax_u + bx_v.$

$S(x_v) = cx_u + dx_v.$

Hence $S(x_u)\cdot x_u = ax_u\cdot x_u$, implying $a = \ell/E$, and so on.

9(b). $F = m = 0$ implies

$$S(x_u) = (\ell/E)x_u \text{ and } S(x_v) = (n/G)x_v.$$

10. $$x_u \times x_v\cdot(S(v) \times v) = \begin{vmatrix} x_u\cdot S(v) & x_u\cdot v \\ x_v\cdot S(v) & x_v\cdot v \end{vmatrix}$$

$$= \begin{vmatrix} v_1\ell + v_2m & v_1E + v_2F \\ v_1m + v_2n & v_1F + v_2G \end{vmatrix}$$

$$= v_1^2(\ell F - mE) + v_1v_2(\ell G - nE) + v_2^2(mG - nF)$$

$$= -\begin{vmatrix} v_2^2 & -v_1v_2 & v_1^2 \\ E & F & G \\ \ell & m & n \end{vmatrix}$$

11. $V = (1 + u^2)^{\frac{1}{2}}x_u \pm (1 + v^2)^{\frac{1}{2}}x_v$

$$\begin{vmatrix} 1 + v^2 & \mp((1 + u^2)(1 + v^2))^{\frac{1}{2}} & 1 + u^2 \\ 1 + v^2 & uv & 1 + u^2 \\ 0 & 1/W & 0 \end{vmatrix} = 0$$

Section V.5

Technique Exercises

Pages 229-232, Exercises 1, 3, 4, 5, 10, 11, 12.

Other Recommended Exercises

Page 230, Exercise 6. This deals with a coordinate description of principal and asymptotic curves.

Page 231, Exercise 8. This extends Exercise V.5.7.

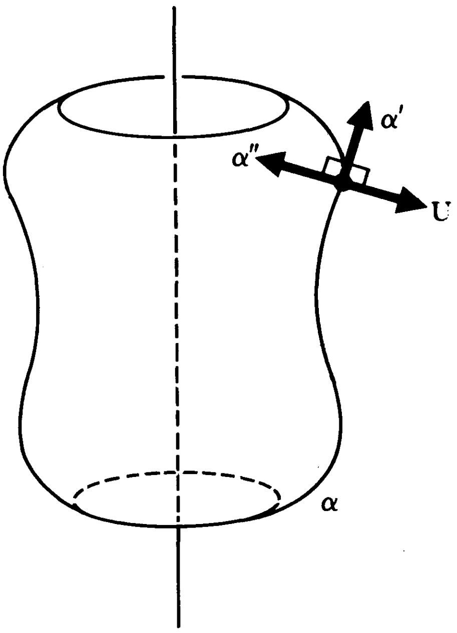
Solutions

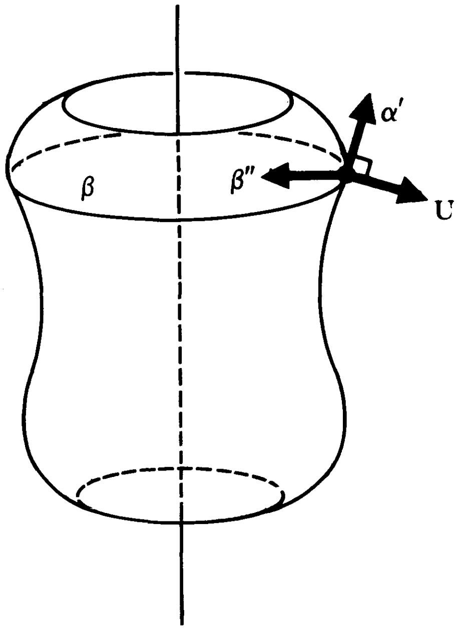
1. α is a straight line $\Rightarrow \alpha'' = 0$

$\Rightarrow \alpha$ is geodesic and asymptotic.

α is geodesic and asymptotic $\Rightarrow \alpha''\cdot U = 0$ and α'' and U are collinear $\Rightarrow \alpha'' = 0$ $\Rightarrow \alpha$ is a straight line.

3.





4. Assume α is unit-speed.

(a) α is asymptotic $\Rightarrow 0 = \alpha''\cdot U = T'\cdot U = \kappa N\cdot U$. Hence T and N are tangents to the surface and B is normal.

So $S(T) = -B' = \tau N$.

(b) Since S is symmetric,

$$S(N) = \tau T + \lambda N$$

for some λ. The matrix for S is $\begin{pmatrix} 0 & \tau \\ \tau & \lambda \end{pmatrix}$

$$K = \det S = -\tau^2.$$

(c) Since $\kappa = 0$ the v-parameter curves $u = u_0$ are asymptotic. A typical one of these is the helix

$$\alpha : v \longmapsto (u_0 \cos v,\ u_0 \sin v,\ bv).$$

$$\tau = b/(b^2 + u_0^2)$$

$$K = -b^2/(b^2 + u_0^2)^2.$$

5. α is principal in M $\Rightarrow k\alpha' = -U'$.

But $U\cdot\overline{U}$ = constant $\Rightarrow U'\cdot\overline{U} + U\cdot\overline{U}' = 0$.

Hence $0 = -k\alpha'\cdot\overline{U} + U\cdot\overline{U}' = U\cdot\overline{U}'$.

Thus $\overline{U}'$ and α' are both orthogonal to U and to $\overline{U}$: if these are linearly independent then α' and $\overline{U}'$ are collinear and so α is principal in $\overline{M}$.

If U and $\overline{U}$ are linearly dependent then $U = \pm\overline{U}$ and so α is principal in M if and only if α is principal in $\overline{M}$.

10. $x_u \times x_v = \beta' \times \delta + v\delta' \times \delta$

$n = 0$

$m = x_{uv} \cdot (x_u \times x_v)/W = \delta' \cdot \beta' \times \delta / W = \beta' \cdot \delta \times \delta' / W$

11(a). Cone: $x = p + v\delta$.

$\beta = p; \beta' = 0; K = 0.$

Cylinder: $x = \beta + vq$.

$\delta = q; \delta' = 0; K = 0.$

11(b). $K = -(\beta' \cdot T \times T')^2/W^4$

$= 0,$ since $\beta' = T$.

12. $K = -(\alpha' \cdot U \times U')/W^4$

$= -(U \cdot U' \times \alpha')/W^4$

$K = 0 \Longleftrightarrow U' \times \alpha' = 0$

$\Longleftrightarrow \alpha'$ and U' are collinear

$\Longleftrightarrow \alpha$ is principal.

6(a). $\alpha' = a_1' x_u + a_2' x_v$. Use Exercise V.4.10.

6(b). $S(\alpha') \cdot \alpha' = \ell a_1'^2 + 2m a_1' a_2' + n a_2'^2$.

8(a). α is principal and geodesic

$\Longleftrightarrow g = t = 0$

$\Longleftrightarrow V =$ constant

$\Longleftrightarrow \alpha$ lies in a plane orthogonal to M along α.

8(b). α is principal and asymptotic

$\Longleftrightarrow t = k = 0$

$\Longleftrightarrow U =$ constant

$\Longleftrightarrow \alpha$ lies in a plane tangent to M along α.

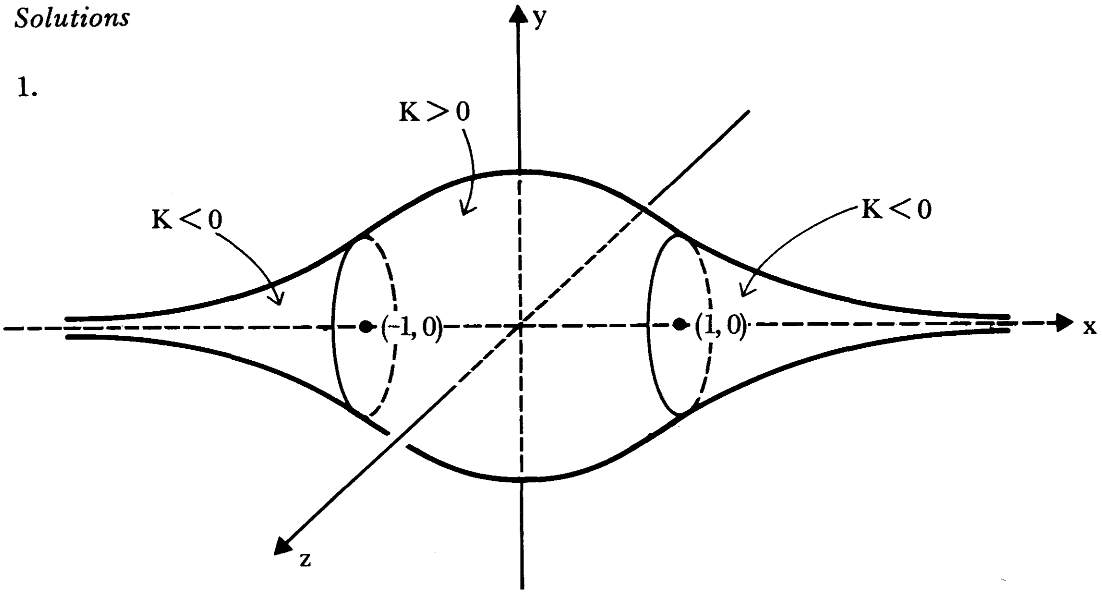
Section V.6

Technique Exercises

Pages 242-243, Exercises 1, 3, 4, 5, 6, 8. (Errata in Exercise 3: in line −3 "h(u)" should be "$h(u)^{-1}$", and in line −1 "h" should be "h^{-1}".

Solutions

1.



$g(u) = u,\ h(u) = \exp(-u^2/2),$

$K(u) = (1 - u^2)/(1 + u^2e^{-u^2})^2.$

3. $\alpha = (g, h, 0)$

$\alpha' = (g', h', 0)$

$\alpha'' = (g'', h'', 0)$

$$\kappa = \|\alpha' \times \alpha''\| / \|\alpha'\|^3 = |g'h'' - h'g''|/(g'^2 + h'^2)^{\frac{3}{2}}$$
$$= |k_\mu|$$
$$|\cos\varphi| = \frac{|g'|}{(g'^2 + h'^2)^{\frac{1}{2}}} = h|k_\pi|$$
$$|K| = |k_\mu||k_\pi| = \frac{\kappa}{h}|\cos\varphi|$$

4. $\alpha(u) = (R + a\cos u,\ b\sin u,\ 0)$

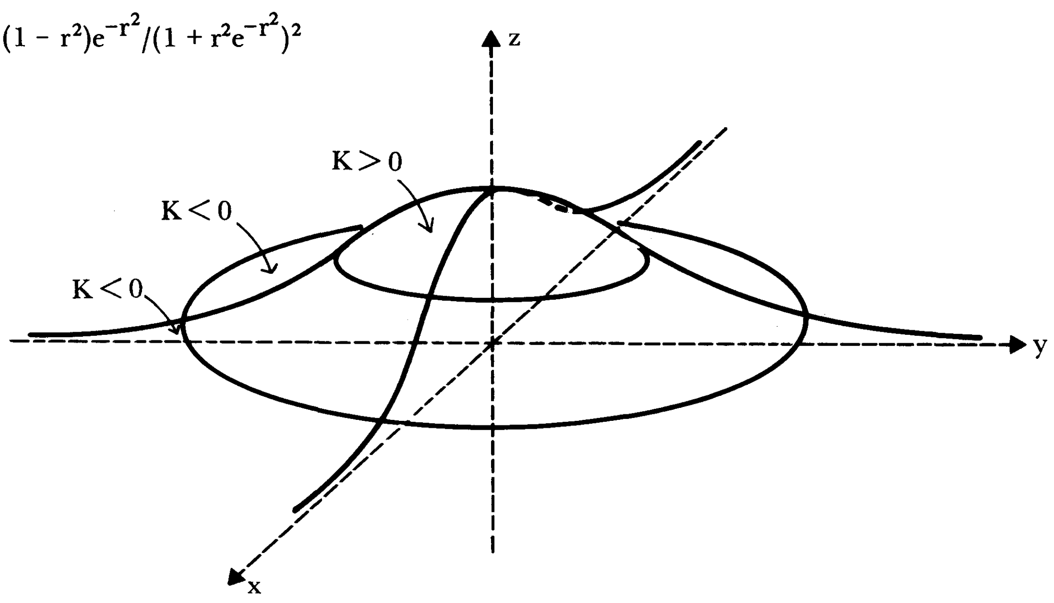
$K(u) = b^2 a\cos u/(R + a\cos u)(b^2\cos^2 u + a^2\sin^2 u)^2$

5. $x(r, v) = (r\cos v,\ r\sin v,\ f(r))$

$g(r) = f(r),\ h(r) = r,$

$K(r) = f'(r)f''(r)/r(1 + f'(r)^2)^2.$

6. $K(r) = (1 - r^2)e^{-r^2}/(1 + r^2e^{-r^2})^2$



8(a). $g(u) = u + \sin u$, $h(u) = 1 + \cos u$,

$K(u) = 1/4(1 + \cos u)$.

8(b). $s(t) = 4 \sin(t/2)$

$h(s) = 2 - s^2/8$

$\overline{K}(s) = 1/4(2 - s^2/8)$.

8(c). $1/4(2 - s^2/8) = 1/4(1 + \cos t)$.

DIFFERENTIAL GEOMETRY

I Calculus on Euclidean Space

II Frame Fields

III Euclidean Geometry

IV Calculus on a Surface

V Shape Operators

VI Geometry of Surfaces in E^3

M334 PART VI

516.
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PART VI GEOMETRY OF SURFACES IN E^3

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Mathematics: A Third Level Course

DIFFERENTIAL GEOMETRY PART VI

GEOMETRY OF SURFACES IN E^3

A commentary on Chapter VI of O'Neill's
Elementary Differential Geometry
Prepared by the Course Team

THE OPEN UNIVERSITY PRESS

M334 VI

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1.2

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Set Book

Barrett O'Neill: *Elementary Differential Geometry* (Academic Press, 1966). It is essential to have this book: the course is based on it and will not make sense without it.

Bibliography

The set books for M201, M231 and MST 282 are referred to occasionally; they are useful but not essential. They are:

D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins: *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

E.D. Nering: *Linear Algebra and Matrix Theory* (John Wiley, 1970).

M. Spivak: *Calculus,* paperback edition, (W.A. Benjamin/Addison-Wesley, 1973).

R.C. Smith and P. Smith: *Mechanics,* SI edition (John Wiley, 1972).

Conventions

Before starting work on this text, please read M334 Part Zero, Consult the Errata List and the Stop Press and make any necessary alterations for this chapter in the set book.

Unreferenced pages and sections denote the set book. Otherwise

O'Neill	denotes the set book;
Text	denotes the correspondence text;
KKOP	denotes *An Introduction to Linear Analysis* by D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins;
Nering	denotes *Linear Algebra and Matrix Theory* by E.D. Nering;
Spivak	denotes *Calculus* by M. Spivak;
Smith	denotes *Mechanics* by R.C. Smith and P. Smith.

References to Open University Courses in Mathematics take the form:

Unit M100 22, Linear Algebra I

Unit MST 281 10, Taylor Approximation

Unit M201 16, Euclidean Spaces I: Inner Products

Unit M231 2, Functions and Graphs

Unit MST 282 1, Some Basic Tools.

VI.1 INTRODUCTION AND THE FUNDAMENTAL EQUATIONS

Introduction

In this chapter we draw together all the results that we have developed in this course. In this section we use frame fields, introduced in Section II.6, their structure equations from Section II.8, the calculus of forms on surfaces from Section IV.4, orientability from Section IV.7 and the theory of the shape operator from Chapter V. You should be familiar with these topics before tackling this section.

In Chapters I and IV we developed two different approaches to differential forms. In Chapter I we introduced forms in E^3 in terms of simple forms such as dx and dx dy, while in Chapter IV we introduced differential forms on a surface in E^3, $M \subset E^3$, in a coordinate independent manner. We showed that the two approaches gave structures with similar properties. In this chapter we want to be able to use both approaches interchangeably. We want to use the structural equations, that were obtained in terms of coordinates in Section II.8, by interpreting the forms as linear functionals and bilinear forms operating on tangent vector fields. For now, accept that the two approaches are equivalent. In the Optional Text we indicate why this is so and give a more abstract approach to some of the results of this section. The Optional Text is not examinable so do not waste any time on it if you find it difficult.

READ: The Introduction to Chapter VI and Section VI.1 (pages 245-250).

Optional Text

It is possible to show that the approach to forms developed in Chapter I is equivalent to that developed in Chapter IV and consequently that we can interpret the forms in Theorem VI.1.7 as linear functionals and bilinear forms. It will be necessary to interpret them this way in subsequent sections.

Rather than prove that the two approaches are equivalent we shall indicate how they both follow from a more abstract approach. The approach of Chapter I is in terms of coordinates and the approach of Chapter IV uses patches; a more abstract approach uses the *Poisson bracket* of vector fields.

For our purposes the most useful definition is given in Exercise 7, page 81. For vector fields V, W the Poisson bracket is the vector field

$$[V, W] = \nabla_V W - \nabla_W V.$$

Part (a) of Exercise 7 tells us that

$$[V, W] = VW - WV,$$

from which we deduce that, given a vector field $Z = (z_1, z_2, z_3)$,

$$\begin{aligned}\nabla_{[V, W]} Z &= \sum_{i=1}^{3} ([V, W][z_i])U_i \\ &= \sum_{i=1}^{3} ((VW - WV)[z_i])U_i \\ &= \Sigma V(W[z_i])U_i - \Sigma W(V[z_i])U_i \\ &= \nabla_V(\Sigma W[z_i]U_i) - \nabla_W(\Sigma V[z_i]U_i) \\ &= \nabla_V(\nabla_W Z) - \nabla_W(\nabla_V Z) \\ &= (\nabla_V \nabla_W - \nabla_W \nabla_V)(Z),\end{aligned}$$

where $\nabla_V\nabla_W$ and $\nabla_W\nabla_V$ represent the appropriate composite mappings.

Hence

$$\nabla_{[V,W]} = \nabla_V\nabla_W - \nabla_W\nabla_V.$$

When we are dealing with a surface M in E^3 we need to know that the Poisson bracket of two tangent vector fields is again a tangent vector field. This is proved in Exercise 9, page 195.

Now we are in a position to give an abstract definition of the differential $d\phi$ of a 1-form ϕ (on a surface). We evaluate the form on a pair of (tangent) vector fields as follows:

$$d\phi(V, W) = V[\phi(W)] - W[\phi(V)] - \phi([V, W]),$$

where here we are using directional derivatives.

Now we can deal with the equations of Theorem VI.1.7 without using Theorem II.8.3. For instance the first equation of part (1) is proved by applying both sides of the equation to the vector fields E_1 and E_2. For example,

$$\begin{aligned} d\theta_1(E_1, E_2) &= E_1[\theta_1(E_2)] - E_2[\theta_1(E_1)] - \theta_1([E_1, E_2]) \\ &= E_1[0] - E_2[1] - \theta_1([E_1, E_2]) \\ &= 0 - \theta_1(\nabla_{E_1}E_2 - \nabla_{E_2}E_1) \\ &= -E_1\cdot\nabla_{E_1}E_2 + E_1\cdot\nabla_{E_2}E_1. \end{aligned}$$

Since $E_1\cdot E_2 = 0$ and $E_1\cdot E_1 = 1$ it follows that

$$E_1[E_1\cdot E_2] = 0 = E_2[E_1\cdot E_1],$$

and from the Leibnizian property of the covariant derivative (Corollary II.5.4, page 80) we can deduce that

$$\nabla_{E_1}E_1\cdot E_2 + E_1\cdot\nabla_{E_1}E_2 = 0$$

and

$$\nabla_{E_2}E_1\cdot E_1 + E_1\cdot\nabla_{E_2}E_1 = 2E_1\cdot\nabla_{E_2}E_1 = 0.$$

Hence

$$\begin{aligned} d\theta_1(E_1, E_2) &= E_2\cdot\nabla_{E_1}E_1 \\ &= \omega_{12}(E_1), \qquad \text{by Lemma II.7.1.} \end{aligned}$$

Also

$$\begin{aligned} (\omega_{12}\wedge\theta_2)(E_1, E_2) &= \omega_{12}(E_1)\theta_2(E_2) - \omega_{12}(E_2)\theta_2(E_1) \\ &= \omega_{12}(E_1). \end{aligned}$$

Equations like part (3) are proved by applying both sides of the equation to the pair of tangent vector fields $\{E_1, E_2\}$. We find

$$\begin{aligned} d\omega_{12}(E_1, E_2) &= E_1[\omega_{12}(E_2)] - E_2[\omega_{12}(E_1)] - \omega_{12}([E_1, E_2]) \\ &= E_1[E_2\cdot\nabla_{E_2}E_1] - E_2[E_2\cdot\nabla_{E_1}E_1] - E_2\cdot\nabla_{[E_1, E_2]}E_1 \\ &= \nabla_{E_1}E_2\cdot\nabla_{E_2}E_1 + E_2\cdot\nabla_{E_1}\nabla_{E_2}E_1 - \nabla_{E_2}E_2\cdot\nabla_{E_1}E_1 \\ &\quad - E_2\cdot\nabla_{E_2}\nabla_{E_1}E_1 - E_2\cdot(\nabla_{E_1}\nabla_{E_2}E_1 - \nabla_{E_2}\nabla_{E_1}E_1) \end{aligned}$$

$$= \nabla_{E_1} E_2 \cdot \nabla_{E_2} E_1 - \nabla_{E_2} E_2 \cdot \nabla_{E_1} E_1$$

$$= (\omega_{21}(E_1)E_1 + \omega_{23}(E_1)E_3) \cdot (\omega_{12}(E_2)E_2 + \omega_{13}(E_2)E_3)$$

$$- (\omega_{21}(E_2)E_1 + \omega_{23}(E_2)E_3) \cdot (\omega_{12}(E_1)E_2 + \omega_{13}(E_1)E_3)$$

$$= \omega_{23}(E_1)\omega_{13}(E_2) - \omega_{23}(E_2)\omega_{13}(E_1)$$

$$= (\omega_{13} \wedge \omega_{32})(E_1, E_2).$$

Now you can see why O'Neill has avoided such proofs, although they can be simplified by careful use of suffices.

It remains to show only that this new approach to the differential of a 1-form includes those of both Chapters I and IV.

How does this give us the formulas of Chapter I? Firstly, in E^3 we notice that, since the standard frame fields are parallel, the Poisson brackets $[U_i, U_j]$ are zero. Hence if $\phi = a_1 dx_1 + a_2 dx_2 + a_3 dx_3$ then

$$d\phi(U_1, U_2) = U_1[\phi(U_2)] - U_2[\phi(U_1)]$$

$$= U_1[a_2] - U_2[a_1]$$

$$= \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}.$$

But, using a definition of the wedge product similar to that used in Chapter IV,

$$dx_1 dx_2 (U_1, U_2) = dx_1(U_1)dx_2(U_2) - dx_1(U_2)dx_2(U_1)$$

$$= 1.1 - 0.0 = 1.$$

Using these and similar results we see that

$$d\phi = \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right) dx_1 dx_2 + \text{two similar terms.}$$

How do we obtain the formula of Definition IV.4.4? Firstly, if we are dealing with a patch x we note that the Poisson bracket $[x_u \circ x^{-1}, x_v \circ x^{-1}]$ of the partial velocity tangent vector fields is zero. This follows since at the point $x(u_0, v_0)$

$$[x_u \circ x^{-1}, x_v \circ x^{-1}]\ (x(u_0, v_0))$$

$$= \nabla_{x_u \circ x^{-1}(x(u_0, v_0))}(x_v \circ x^{-1}) - \nabla_{x_v \circ x^{-1}(x(u_0, v_0))}(x_u \circ x^{-1})$$

$$= \nabla_{x_u(u_0, v_0)}(x_v \circ x^{-1}) - \nabla_{x_v(u_0, v_0)}(x_u \circ x^{-1})$$

$= x_{vu}(u_0, v_0) - x_{uv}(u_0, v_0)$, as in Comment (i) on Section V.4,

$= 0$, since $x_{uv} = x_{vu}$.

Hence

$$d\phi(x_u \circ x^{-1}, x_v \circ x^{-1}) = x_u \circ x^{-1}[\phi(x_v \circ x^{-1})] - x_v \circ x^{-1}[\phi(x_u \circ x^{-1})]$$

and hence considered as functions on the domain of the patch x

$$d\phi(x_u, x_v) = x_u[\phi(x_v \circ x^{-1})] - x_v[\phi(x_u \circ x^{-1})]$$

$$= \frac{\partial}{\partial u}[\phi(x_v \circ x^{-1} \circ x)] - \frac{\partial}{\partial v}[\phi(x_u \circ x^{-1} \circ x)]$$

$$= \frac{\partial}{\partial u}(\phi(x_v)) - \frac{\partial}{\partial v}(\phi(x_u)).$$

Summary

Notation

$\omega_{12}, \omega_{13}, \omega_{23}$	Page 248, line 3
θ_1, θ_2	Page 248, line -15

Definitions

(i)	Adapted frame field	Page 246, Definition 1.1
(ii)	Connection forms $(\omega_{12}, \omega_{13}, \omega_{23})$ on a surface	Page 248, line 3
(iii)	Dual 1-forms θ_1, θ_2 ·	Page 248, line -16

Results

(i)	There is an adapted frame field on a region of M if and only if the region is orientable and there exists a nonvanishing tangent vector field on it. This is always possible on the image of a patch.	Page 246, Lemma 1.2
(ii)	For a tangent vector v tangent to M at p and S the shape operator $S(v) = \omega_{13}(v)E_1(p) + \omega_{23}(v)E_2(p).$	Page 248, Corollary 1.5
(iii)	The structural equations (1) $d\theta_1 = \omega_{12} \wedge \theta_2$, $d\theta_2 = \omega_{21} \wedge \theta_1$ First structural equations (2) $\omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2 = 0$ Symmetry equation (3) $d\omega_{12} = \omega_{13} \wedge \omega_{32}$ Gauss equation (4) $d\omega_{13} = \omega_{12} \wedge \omega_{23}$, $d\omega_{23} = \omega_{21} \wedge \omega_{13}$ Codazzi equations	Page 249, Theorem 1.7

Techniques

(i)	Describing the shape operator in terms of the connection forms.	Page 248, Corollary 1.5
(ii)	Using the structural equations.	Page 249, Theorem 1.7

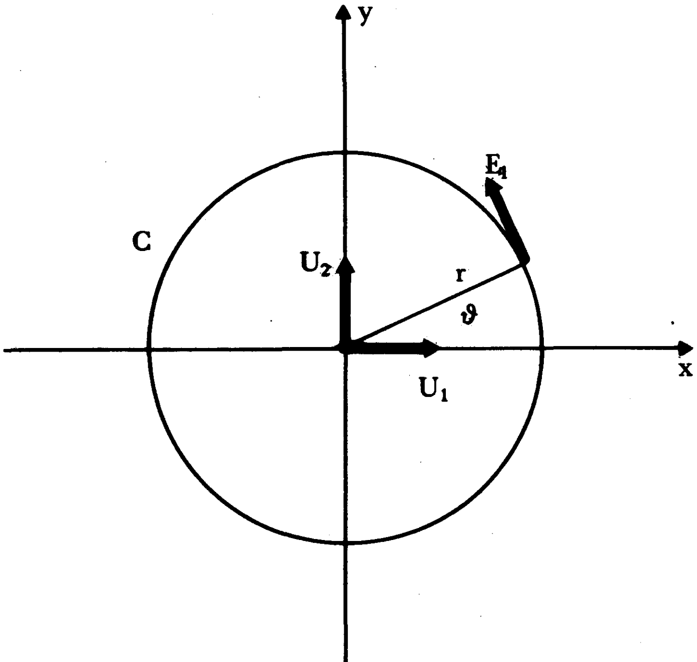
Exercises

Technique (i)

1. Page 250, Exercise 2.

Technique (ii)

2. (a) Let C be a circle of radius r in the xy plane and let the radius make an angle ϑ with the x axis. Let E_1 be the unit vector field along C that is in the xy plane, is tangent to C at each point of C and points in an anti-clockwise direction.



Prove that $E_1[\vartheta] = 1/r$.

(b) Page 250, Exercise 3. (Use the result of part (a) and compare the attitude matrix with that of the frame field of Fig. 6.3 on page 247 and Example 1.6 on page 248.)

(c) Page 250, Exercise 4.

Theory Exercise

3. Page 250, Exercise 1.

Solutions

1. Page 250, Exercise 2.

(a) We verify each of the structural equations in turn.

(1) $d\theta_1 = d(r\cos\varphi\, d\vartheta) = -r\sin\varphi\, d\varphi\, d\vartheta$, since $d(d\vartheta) = 0$ and r is constant on the sphere. Also

$\omega_{12} \wedge \theta_2 = \sin\varphi\, d\vartheta \wedge r d\varphi = r\sin\varphi\, d\varphi\, d\vartheta$.

$d\theta_1 = d(r\, d\varphi) = 0$, since r is constant, and

$\omega_{21} \wedge \theta_1 = -\sin\varphi\, d\vartheta \wedge r\cos\varphi\, d\vartheta = 0$, since $d\vartheta \wedge d\vartheta = 0$.

(2) $\omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2$

$= (\cos\varphi\, d\vartheta) \wedge (r\cos\varphi\, d\vartheta) + d\varphi \wedge (r\, d\varphi) = 0.$

(3) $d\omega_{12} = d(\sin\varphi\, d\vartheta) = \cos\varphi\, d\varphi\, d\vartheta$;

$\omega_{13} \wedge \omega_{32} = (-\cos\varphi\, d\vartheta) \wedge d\varphi = \cos\varphi\, d\varphi\, d\vartheta$.

(4) $d\omega_{13} = d(-\cos\varphi d\vartheta) = \sin\varphi\, d\varphi\, d\vartheta;$

$\omega_{12} \wedge \omega_{23} = (\sin\varphi\, d\vartheta) \wedge (-d\varphi) = \sin\varphi\, d\varphi\, d\vartheta.$

$d\omega_{23} = d(-d\varphi) = 0;$

$\omega_{21} \wedge \omega_{13} = (-\sin\varphi\, d\vartheta) \wedge (-\cos\varphi\, d\vartheta) = 0.$

(b) $$E_1[\vartheta] = d\vartheta[E_1] = \frac{1}{r\cos\varphi}\theta_1[E_1] = \frac{1}{r\cos\varphi};$$

$$E_2[\vartheta] = d\vartheta[E_2] = \frac{1}{r\cos\varphi}\theta_1[E_2] = 0;$$

$$E_1[\varphi] = d\varphi[E_1] = \frac{1}{r}\theta_2[E_1] = 0;$$

$$E_2[\varphi] = d\varphi[E_2] = \frac{1}{r}\theta_2[E_2] = \frac{1}{r}.$$

(c) By Corollary VI.1.5,

$$\begin{aligned} S(v) &= \omega_{13}(v)E_1(p) + \omega_{23}(v)E_2(p) \\ &= -(\cos\varphi\, d\vartheta(v)E_1(p) + d\varphi(v)E_2(p)). \end{aligned}$$

Now if

$v = v_1 E_1(p) + v_2 E_2(p)$ then

$$\begin{aligned} d\vartheta(v) &= v_1 d\vartheta(E_1(p)) + v_2 d\vartheta(E_2(p)) \\ &= v_1 E_1(p)[\vartheta] + v_2 E_2(p)[\vartheta] \\ &= v_1/r\cos\varphi(p), \qquad \text{by part (b);} \end{aligned}$$

similarly

$$\begin{aligned} d\varphi(v) &= v_1 E_1(p)[\varphi] + v_2 E_2(p)[\varphi] \\ &= v_2/r. \end{aligned}$$

Hence

$$\begin{aligned} S(v) &= -\left(\frac{v_1\cos\varphi(p)}{r\cos\varphi(p)}E_1(p) + \frac{v_2}{r}E_2(p)\right) \\ &= -\frac{1}{r}(v_1 E_1(p) + v_2 E_2(p)) = -\frac{v}{r}. \end{aligned}$$

Hence the shape operator is given by the matrix

$$S = -\frac{1}{r}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2(a). We can parametrize the circle C in such a way that the initial velocity is E_1 evaluated at the point corresponding to angle ϑ_0; that is,
$E_1 = -\sin\vartheta_0\, U_1 + \cos\vartheta_0\, U_2$.

We need

$$\alpha : t \longmapsto r\cos(\vartheta_0 + \tfrac{t}{r})\, U_1 + r\sin(\vartheta_0 + \tfrac{t}{r})\, U_2.$$

Hence

$$\begin{aligned} E_1[\vartheta] &= \left.\frac{d}{dt}(\vartheta(\alpha))\right|_{t=0} \\ &= \left.\frac{d}{dt}(\vartheta_0 + \frac{t}{r})\right|_{t=0} = \frac{1}{r}. \end{aligned}$$

2(b). Page 250, Exercise 3.

Replacing r by $(R + r\cos\varphi)$ in the previous part of the exercise we find that

$$E_1[\vartheta] = 1/(R + r\cos\varphi).$$

Since E_2 lies in a plane ϑ = constant

$$E_2[\vartheta] = 0.$$

Hence $$d\vartheta[E_1] = 1/(R + r\cos\varphi)$$

$$d\vartheta[E_2] = 0$$

and consequently $$\theta_1 = (R + r\cos\varphi)\,d\vartheta.$$

Similarly $$\theta_2 = r\,d\varphi.$$

Comparing Fig. 6.4 with Fig. 6.3 on page 247 we see that the attitude matrix for the frame field on the torus is the same as that for the frame field on the sphere. Hence the connection forms are the same. Those for the sphere were obtained on page 249 and so

$$\omega_{12} = \sin\varphi\, d\vartheta, \quad \omega_{13} = -\cos\varphi\, d\vartheta, \quad \omega_{23} = -d\varphi.$$

The structural equations are verified as for the spherical frame field.

2(c). Page 250, Exercise 4.

$$S(V) = \omega_{13}(V)E_1 + \omega_{23}(V)E_2.$$

$$S(V) = -\cos\varphi\, d\vartheta(V)E_1 - d\varphi(V)E_2$$

$$= -\frac{\cos\varphi}{(R + r\cos\varphi)}\theta_1(V)E_1 - \frac{1}{r}\theta_2(V)E_2.$$

$$S(E_1) = -(\cos\varphi/(R + r\cos\varphi))\,E_1.$$

$$S(E_2) = -(1/r)E_2.$$

Hence $k_1 = -\cos\varphi/(R + r\cos\varphi)$ and $k_2 = -1/r$.

3. Page 250, Exercise 1.

(a) $$\alpha'' = T' = \nabla_T T, \quad \text{since } T = \alpha',$$

$$= \nabla_T E_1, \quad \text{since } T = E_1,$$

$$= \omega_{12}(T)E_2 + \omega_{13}(T)E_3.$$

Now α is a geodesic if and only if α'' is normal to the surface, which happens if and only if $\omega_{12}(T) = 0$.

(b) If we let T, V, U be the restrictions of E_1, E_2, E_3 to α then

$$T' = \omega_{12}(T)V + \omega_{13}(T)U,$$

and so by Exercise V.5.7

$$g = \omega_{12}(T) \text{ and } k = \omega_{13}(T).$$

Again from Exercise V.5.7

$$t = S(T)\cdot V$$

$$= (\omega_{13}(T)T + \omega_{23}(T)V)\cdot V, \qquad \text{by Corollary 1.5,}$$

$$= \omega_{23}(T).$$

VI.2 FORM COMPUTATIONS

Introduction

This section follows on directly from the previous section. It uses the structural equations to obtain formulas for the mean and Gaussian curvature, introduced in Section V.3, and derives another form of the Codazzi equations when an adapted frame field can be chosen using principal directions.

The equation involving the Gaussian curvature is of fundamental importance in establishing, in Section VI.5, that Gaussian curvature belongs to an 'intrinsic' geometry.

READ: Section VI.2 (pages 251-255).

Supplementary Comments

(i) *Page 252: Lemma 2.2(2), the trace formula*

$$(\omega_{13} \wedge \theta_2 + \theta_1 \wedge \omega_{23})(E_1, E_2)$$

$$= \omega_{13}(E_1)\theta_2(E_2) - \omega_{13}(E_2)\theta_2(E_1) + \theta_1(E_1)\omega_{23}(E_2) - \theta_1(E_2)\omega_{23}(E_1)$$

$$= \omega_{13}(E_1) + \omega_{23}(E_2), \qquad \text{since } \theta_i(E_j) = \delta_{ij},$$

$$= \text{trace } S = 2H,$$

since the trace is found by adding the elements on the diagonal of any matrix representing the linear transformation.

(ii) *Page 253: Corollary 2.4, line -4 of the proof*

$$d\omega_{12}(E_1, E_2) = (df_1 \wedge \theta_1 + df_2 \wedge \theta_2 + f_1\omega_{12} \wedge \theta_2 + f_2\omega_{21} \wedge \theta_1)(E_1, E_2)$$

$$= df_1(E_1)\theta_1(E_2) - df_1(E_2)\theta_1(E_1) + df_2(E_1)\theta_2(E_2) - df_2(E_2)\theta_2(E_1)$$

$$+ f_1\omega_{12}(E_1)\theta_2(E_2) - f_1\omega_{12}(E_2)\theta_2(E_1) + f_2\omega_{21}(E_1)\theta_1(E_2) - f_2\omega_{21}(E_2)\theta_1(E_1)$$

$$= -df_1(E_2) + df_2(E_1) + f_1\omega_{12}(E_1) - f_2\omega_{21}(E_2).$$

(iii) *Page 253: line -7*

$$\omega_{12}(E_1) = \sin\varphi\, d\vartheta(E_1) = \sin\varphi E_1[\vartheta] = \sin\varphi/r\cos\varphi, \qquad \text{by Exercise VI.1.2,}$$

$$= \tan\varphi/r.$$

$$\omega_{12}(E_2) = \sin\varphi E_2[\vartheta] = 0.$$

(iv) *Page 253: line -3* This uses the chain rule $V[h(f)] = h'(f)\, V[f]$, where V is a tangent vector field on M, f is a function from M to R and h is a function from R to itself.

(v) *Page 254: Lemma 2.6, line 7 of the proof* If $V_1 = aF_1 + bF_2$ is an eigenvector corresponding to the eigenvalue k_1 then

$$\begin{pmatrix} S_{11} - k_1 & S_{12} \\ S_{12} & S_{22} - k_1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so

$$a(S_{11} - k_1) + bS_{12} = 0.$$

Choosing $a = S_{12}$ gives $b = k_1 - S_{11}$ and so V_1 is as stated; and V_2 follows similarly.

(vi) *Page 254: Lemma 2.6, line −5 of the proof* $S(F_1) \cdot F_2$ is zero if and only if $S(F_1)$ is perpendicular to F_2, that is collinear to F_1. Hence $S(F_1) \cdot F_2$ is zero if and only if F_1 is a principal vector, which possibility we have excluded by considering only a small neighbourhood about p.

Summary

Definitions

(i)	Tangent frame field	Page 251, Section VI.2, line 4
(ii)	Principal frame field	Page 254, Definition 2.5

Results

(i)	For 1-forms ϕ and ψ and 2-forms μ and ν, $\phi = \psi$ if and only if $\phi(E_1) = \psi(E_1)$ and $\phi(E_2) = \psi(E_2)$ and $\mu = \nu$ if and only if $\mu(E_1, E_2) = \nu(E_1, E_2)$.	Page 251, lines −13 and −11
(ii)	$\theta_i(E_j) = \delta_{ij}$.	Page 251, line −7
(iii)	$\phi = \phi(E_1)\theta_1 + \phi(E_2)\theta_2$. $\mu = \mu(E_1, E_2)\theta_1 \wedge \theta_2$.	Page 251, Lemma 2.1
(iv)	$\omega_{13} \wedge \omega_{23} = K\theta_1 \wedge \theta_2$. $\omega_{13} \wedge \theta_2 + \theta_1 \wedge \omega_{23} = 2H\theta_1 \wedge \theta_2$.	Page 252, Lemma 2.2
(v)	$d\omega_{12} = -K\theta_1 \wedge \theta_2$.	Page 252, Corollary 2.3
(vi)	$K = E_2[\omega_{12}(E_1)] - E_1[\omega_{12}(E_2)] - \omega_{12}(E_1)^2 - \omega_{12}(E_2)^2$.	Page 253, Corollary 2.4
(vii)	If p is nonumbilic then there exists a principal frame field on some neighbourhood of p in M.	Page 254, Lemma 2.6
(viii)	If E_1, E_2, E_3 is a principal frame field then $\omega_{13} = k_1\theta_1, \quad \omega_{23} = k_2\theta_2$.	Page 254, line −4
(ix)	If E_1, E_2, E_3 is a principal frame field then $E_1[k_2] = (k_1 - k_2)\,\omega_{12}(E_2)$ $E_2[k_1] = (k_1 - k_2)\,\omega_{12}(E_1)$.	Page 255, Lemma 2.7

Techniques

(i)	Expressing forms in terms of θ_1, θ_2 and $\theta_1 \wedge \theta_2$.	Page 251, Lemma 2.1
(ii)	Using the structural equations to evaluate exterior derivatives of forms described in terms of $\theta_1, \theta_2, \omega_{12}, \omega_{13}$ and ω_{23}.	Page 249, Theorem 1.7
(iii)	Using the second structural equation to find the Gaussian curvature: $d\omega_{12} = -K\theta_1 \wedge \theta_2$.	Page 252, Corollary 2.3

Exercises

Techniques (i) and (ii)

1. Page 256, Exercise 2.
2. Page 256, Exercise 3.

Technique (iii)

3. Use the formulas of Exercise 3 on page 250 to calculate the Gaussian curvature of the torus.

Solutions

1. Page 256, Exercise 2.

Since $h_i = X \cdot E_i$, then for any vector field V on M

$$\begin{aligned} dh_i(V) &= V[h_i] = V[X \cdot E_i] \\ &= \nabla_V X \cdot E_i + X \cdot \nabla_V E_i, \qquad \text{by Corollary II.5.4(4),} \\ &= V \cdot E_i + X \cdot \nabla_V E_i. \end{aligned}$$

Hence, by the definition of the dual 1-forms and the connection forms,

$$\begin{aligned} dh_1(V) &= \theta_1(V) + X \cdot (\omega_{12}(V)E_2 + \omega_{13}(V)E_3) \\ &= \theta_1(V) + \omega_{12}(V)\,h_2 + \omega_{13}(V)\,h_3. \end{aligned}$$

Since this is true for all tangent vector fields V it follows that

$$dh_1 = \theta_1 + \omega_{12}h_2 + \omega_{13}h_3$$

and similarly

$$dh_2 = \theta_2 + \omega_{21}h_1 + \omega_{23}h_3.$$

2. Page 256, Exercise 3.

(a) By definition $\psi(V) = X \cdot V \times E_3$.

Now
$$\begin{aligned} \psi &= \psi(E_1)\theta_1 + \psi(E_2)\theta_2 \\ &= (X \cdot (E_1 \times E_3))\theta_1 + (X \cdot (E_2 \times E_3))\theta_2 \\ &= -(X \cdot E_2)\theta_1 + (X \cdot E_1)\theta_2 \\ &= -h_2\theta_1 + h_1\theta_2. \end{aligned}$$

Hence

$$\begin{aligned} d\psi &= -dh_2 \wedge \theta_1 - h_2\,d\theta_1 + dh_1 \wedge \theta_2 + h_1\,d\theta_2 \\ &= -(\theta_2 + \omega_{21}h_1 + \omega_{23}h_3) \wedge \theta_1 - h_2\omega_{12} \wedge \theta_2 \\ &\quad + (\theta_1 + \omega_{12}h_2 + \omega_{13}h_3) \wedge \theta_2 + h_1\,\omega_{21} \wedge \theta_1, \qquad \text{by Exercise 1,} \\ &= 2\theta_1 \wedge \theta_2 + h_3(\omega_{13} \wedge \theta_2 + \theta_1 \wedge \omega_{23}) \\ &= 2\theta_1 \wedge \theta_2 + 2h_3H\theta_1 \wedge \theta_2, \qquad \text{by Lemma VI.2.2(2),} \\ &= 2(1 + hH)\theta_1 \wedge \theta_2. \end{aligned}$$

where $h_3 = h$ is the support function.

(b) $$\begin{aligned} \zeta(V) &= \psi(S(V)) \\ &= \psi(\omega_{13}(V)E_1 + \omega_{23}(V)E_2), \qquad \text{by Corollary VI.1.5,} \\ &= X\cdot(\omega_{13}(V)E_1 + \omega_{23}(V)E_2)\times E_3 \\ &= -\omega_{13}(V)(X\cdot E_2) + \omega_{23}(V)(X\cdot E_1). \end{aligned}$$

Hence

$$\zeta = -h_2\,\omega_{13} + h_1\,\omega_{23}$$

and

$$\begin{aligned} d\zeta &= -dh_2 \wedge \omega_{13} - h_2\,d\omega_{13} + dh_1 \wedge \omega_{23} + h_1\,d\omega_{23} \\ &= -(\theta_2 + \omega_{21}h_1 + \omega_{23}h_3)\wedge\omega_{13} - h_2(\omega_{12}\wedge\omega_{23}) \\ &\quad + (\theta_1 + \omega_{12}h_2 + \omega_{13}h_3)\wedge\omega_{23} + h_1(\omega_{21}\wedge\omega_{13}), \qquad \text{by Exercise 1,} \\ &= \omega_{13}\wedge\theta_2 + \theta_1\wedge\omega_{23} + 2h_3(\omega_{13}\wedge\omega_{23}) \\ &= 2(H + hK)\theta_1 \wedge \theta_2, \qquad \text{by Lemma VI.2.2 (1) and (2),} \end{aligned}$$

where $h_3 = h$ is the support function.

3. $d\omega_{12} = d(\sin\varphi\, d\vartheta) = \cos\varphi\, d\varphi\, d\vartheta$;

$$\begin{aligned} \theta_1\wedge\theta_2 &= r(R + r\cos\varphi)\, d\vartheta\, d\varphi \\ &= -r(R + r\cos\varphi)\, d\varphi\, d\vartheta. \end{aligned}$$

Since $d\omega_{12} = -K\theta_1\wedge\theta_2$ it follows that

$$K = \frac{\cos\varphi}{r(R + r\cos\varphi)}\,.$$

VI.3 SOME GLOBAL THEOREMS

Introduction

This section is of a rather different nature to those we have just studied. In this section we study how the shape operator determines the shape of the surface in some restricted cases:

> If the shape operator is identically zero the surface belongs to a plane;
>
> If the surface is all umbilic then it is part of a plane or a sphere.

To go further you will need to be familiar with the concepts of *compactness* and *connectedness*. If you have not met compactness do not worry since the work depending on it is contained in the Optional Text. If you have met compactness and connectedness you should find these results interesting.

Connectedness and Compactness were defined in M202 in *Mendelson: Introduction To Topology,* Definition IV.2.2 and Definition V.2.4 respectively. In M332 they are given by the Definitions on pages 77 and 89 of *Unit 2, Continuous Functions.*

You will need to use the following two results:

(1) If a compact surface M is contained in a connected surface N then M = N.

(Sketch of proof: Since M is compact it is closed. Since M is a surface each point has a neighbourhood homeomorphic to an open disc and hence it is open in the subspace topology. The result now follows from the definition of connectedness.)

(2) If f is a continuous real valued function on a compact surface then f is bounded and attains its bounds.

(This was proved in *Unit M332 2, Continuous Functions,* pages 90-92, Theorems 10-12. It also appears as *Mendelson* Theorems V.2.9 and V.3.5.)

With these results you can read the Optional Text, where we prove the following important theorem:

> If the Gaussian curvature is constant and the surface is compact then the surface is a sphere.

READ: Section VI.3 (page 256 - page 259, line 7).

Comment

(i) *Page 258: line 3* If α is any curve in M then

$$dK(\alpha') = \alpha'[K] = (K \circ \alpha)' = 0, \qquad \text{by Lemma I.4.6.}$$

Hence $K \circ \alpha$ is constant; that is, K is constant along the curve α. Since the surface is connected any two points can be connected by a curve and hence the Gaussian curvature K is the same at any point of the surface.

Supplementary Comments

(i) *Page 256 : line -2* For any curve α, $S(\alpha') = -U'$, where the normal vector field U is restricted to α. In this case $U = E_3$ and $S(\alpha') = 0$. Hence $E_3' = 0$ and so the vector field E_3 is parallel along α. Since M is connected any two points can be connected by a curve and so E_3 is parallel on all of M.

(ii) *Page 256: line -1* The plane through p orthogonal to the direction E_3 consists of all those points r such that $(r - p) \cdot E_3 = 0$.

(iii) *Page 257: line 6* $\alpha' \cdot E_3 = 0$ since α' is a tangent vector and E_3 is the unit normal.

(iv) *Page 257: line −2* Theorem VI.2.7 gives

$$E_1[k] = E_1[k_2] = (k_1 - k_2)\,\omega_{12}(E_2)$$
$$= (k-k)\omega_{12}(E_2) = 0,$$

and similarly $E_2[k] = 0$.

(v) *Page 258: line 1* $dk = dk(E_1)\theta_1 + dk(E_2)\theta_2 = 0.$

(vi) *Page 259: the end of the proof of Theorem 3.3* If we have chosen the outward normal for E_3 it follows that k is the negative square root of K, while if we have chosen the inward normal for E_3 it follows that k is the positive square root of K.

Optional Text

READ: Section VI.3 (page 259, line 8 to page 263).

Supplementary Comments

(i) *Page 262: first paragraph of the proof of Theorem 3.7*

$$H^2 - K = \left(\frac{k_1 + k_2}{2}\right)^2 - k_1k_2 = \frac{k_1^2 + 2k_1k_2 + k_2^2 - 4k_1k_2}{4}$$
$$= \frac{k_1^2 - 2k_1k_2 + k_2^2}{4} = \frac{(k_1 - k_2)^2}{4} \geqslant 0.$$

$H^2 - K = 0 \Rightarrow k_1 = k_2$ and so M is all-umbilic.

(ii) *Page 262: last four lines of the proof of Theorem 3.7*

$$(k_1 + k_2)^2 = (k_1 - k_2)^2 + 4K.$$

Suppose $k_1 > k_2 > 0$: then since K is constant $k_1 + k_2$ and $k_1 - k_2$ attain local maxima together, at which points $k_1 = \frac{1}{2}((k_1 + k_2) + (k_1 - k_2))$ does also. Since $k_2 = K/k_1$ the value of k_2 is a local minimum there.

Summary

Results

(i)	If the shape operator is identically zero then M is part of a plane.	Page 256, Theorem 3.1
(ii)	If M is all-umbilic it has constant Gaussian curvature $K \geqslant 0$.	Page 257, Lemma 3.2
(iii)	If M is all-umbilic and $K > 0$ then M is part of a sphere of radius $1/\sqrt{K}$.	Page 258, Theorem 3.3
(iv)	A compact all-umbilic surface is an entire sphere.	Page 259, Corollary 3.4
(v)	If M is a compact surface there is a point at which the Gaussian curvature K is strictly positive.	Page 259, Theorem 3.5
(vi)	If M is compact with constant Gaussian curvature K, then M is a sphere of radius $1/\sqrt{K}$.	Page 262, Theorem 3.7

Technique

(i) Application of the above results.

Exercise

Technique (i)

1. Page 263, Exercise 1.

Solution

1. Page 263, Exercise 1.

Since M is flat and minimal

$$K = k_1 k_2 = 0 \quad \text{and} \quad H = \tfrac{1}{2}(k_1 + k_2) = 0$$

and so $k_1 = k_2 = 0$. Hence $S = 0$ and M is part of a plane.

VI.4 ISOMETRIES AND LOCAL ISOMETRIES

Introduction

In Chapter III we defined the isometries of E^3 to be the distance-preserving mappings of E^3, and said that the geometry of E^3 consisted of those properties invariant under isometries of E^3. In order to make sense of the geometry of a surface we need the concept of "isometry of surfaces", and this section is devoted to adapting the ideas of Sections III.1 and III.2, that is the definition of isometries and the fact that their derivative maps preserve dot products of tangent vectors, to provide a definition of "isometry of surfaces" and explore some of its consequences.

Take care not to confuse "isometry of surfaces" with "isometry of E^3". An isometry of E^3 is a mapping from E^3 to E^3 which preserves lengths; an isometry of surfaces is a mapping F from a surface M to a surface N with certain nice properties : in general it is not possible to extend F so that it is defined on the whole of E^3, and so F cannot be the restriction of an isometry of E^3. The exact relationship between these two sorts of isometry will be explained in Section VI.8.

Of course, since isometries of surfaces are firstly mappings of surfaces, we need the results of Section IV.5 and the comments thereon. All that is needed from subsequent sections of O'Neill are the definitions of E, F and G on page 210 and the canonical parametrization of the catenoid from Section V.6. O'Neill also refers forward to Exercise 1 on page 269: this is a rewording of the following theorem of linear algebra, which is very similar to many results we used in Chapter III.

Theorem VI.4.A If V and W are 2-dimensional real vector spaces with inner products, which are written with dots in the usual way, and $T : V \longrightarrow W$ is a linear transformation, then the following conditions on T are logically equivalent; that is, if any one of them is true then all the rest are also true.

(a) T preserves dot products; that is,

$$T(v_1)\cdot T(v_2) = v_1 \cdot v_2$$

for all v_1, v_2 in V.

(b) T preserves lengths; that is,

$$\|T(v)\| = \|v\|$$

for all v in V.

(c) T preserves orthonormal bases; that is, if e_1, e_2 is any orthonormal basis for V then $T(e_1), T(e_2)$ is an orthonormal basis for W.

(c′) T preserves a particular orthonormal basis; that is, there is some particular orthonormal basis e_1, e_2 for V such that $T(e_1), T(e_2)$ is an orthonormal basis for W.

(d) There is some pair of linearly independent vectors v_1, v_2 in V such that

$$\|T(v_1)\| = \|v_1\|$$
$$\|T(v_2)\| = \|v_2\|$$
$$T(v_1)\cdot T(v_2) = v_1 \cdot v_2 .$$

Proof Conditions (a) and (b) were shown to be equivalent in *Unit M201 24, Orthogonal and Symmetric Transformations,* Theorem 1 (page 7) and the details of the proof of Theorem 2 on pages 10-11 of that unit show that conditions (a), (c) and (c′) are all equivalent. The remainder of the proof is contained in the following exercise.

1. Prove that (c′) ⇒ (d) and (d) ⇒ (b), where (b), (c′) and (d) denote the corresponding conditions of Theorem VI.4.A. This is sufficient to prove that condition (d) is equivalent to each of conditions (a), (b), (c), (c′).

We shall also need

Theorem VI.4.B If $T: V \longrightarrow W$ is an orthogonal transformation then T has an inverse $T^{-1} : W \longrightarrow V$, which is also orthogonal.

Proof The proof is given in the Correspondence Text, Part III, page 6, Comment (iii).

READ: Section VI.4 (pages 263-269).

Comments

(i) *Page 264: Definition 4.1* A curve segment α from p to q is a portion of a curve defined on a *closed* interval [a, b] such that $\alpha(a) = p$, $\alpha(b) = q$. The length of such a curve segment is the arc length from a to b, that is

$$\int_a^b \|\alpha'(t)\| \, dt.$$

Because all such lengths are non-negative, the set of appropriate "L(α)" s is bounded below (by zero) and so does have a greatest lower bound.

(ii) *Page 264: Definition 4.2* We require that

$$F_*(v) \cdot F_*(w) = v \cdot w$$

whenever these expressions are meaningful. $F_*(v)$ and $F_*(w)$ are defined only when v and w are tangent vectors which are tangent to M. If v and w are two such vectors, $v \cdot w$ is defined if and only if v, w belong to the same tangent space $T_p(M)$: in this case $F_*(v)$ and $F_*(w)$ both belong to the tangent space $T_{F(p)}(M)$ and so $F_*(v) \cdot F_*(w)$ is defined. Thus we are requiring that

$$F_*(v) \cdot F_*(w) = v \cdot w$$

whenever v, w belong to the same tangent space $T_p(M)$ for some p in M.

Supplementary Comments

(i) *Page 265: line 1* This is the implication (a) ⇒ (b) of Theorem VI.4.A.

(ii) *Page 265: proof of Theorem 4.3* $F_*(\alpha') = (F(\alpha))'$ by the definition of F_*.

(iii) *Page 266: Lemma 4.5, and subsequently* O'Neill is using the letter F to denote both a mapping of surfaces and the function $x_u \cdot x_v$. This looks confusing, especially where both usages appear in close proximity, but it should always be clear from the context which meaning is intended.

(iv) *Page 266: proof of Lemma 4.5* This proof uses the equivalence (d) ⇔ (a) of Theorem VI.4.A, the fact that x_u and x_v are linearly independent at each point, and the result that

$$F_*(x_u) = (F(x))_u, \quad F_*(x_v) = (F(x))_v,$$

which was proved on page 161.

(v) *Page 267: Example 4.6(1)* This uses the equivalence (c′) ⇔ (a) of Theorem VI.4.A, as $U_1(p)$ and $U_2(p)$ form an orthonormal basis for $T_p(E^2)$.

(vi) *Page 267: Example 4.6(2)* E, F, G (for the helicoid with $b = 1$) were computed in Example V.4.3(1) on page 213; $\bar{E}, \bar{F}, \bar{G}$ (for the catenoid with a canonical parametrization) were computed in Lemma V.6.3 on page 238.

Summary

Notation

$L(\alpha)$	Page 264, Definition 4.1
$\rho(p, q)$	Page 264, Definition 4.1

Definitions

(i)	Length ($L(\alpha)$) of a curve segment α	Text, page 20, Comment (i)
(ii)	Intrinsic distance $\rho(p, q)$	Page 264, Definition 4.1
(iii)	Isometry of surfaces	Page 264, Definition 4.2
(iv)	Isometric surfaces	Page 265, line −8
(v)	Local isometry	Page 265, Definition 4.4
(vi)	Conformal mapping of surfaces	Page 268, Definition 4.7

Examples

(i)	A plane is locally isometric to a circular cylinder.	Page 267, Example 4.6(1)
(ii)	A helicoid is locally isometric to a catenoid.	Page 267, Example 4.6(2)

Results

(i)	Isometries of surfaces have inverse mappings; that is, they are diffeomorphisms.	Page 265, line 6
(ii)	Isometries of surfaces preserve intrinsic distance.	Page 265, Theorem 4.3
(iii)	Isometries preserve the speed and length of curves.	Page 265, line 14
(iv)	A local isometry is an isometry on sufficiently small neighbourhoods.	Page 266, line 7
(v)	A mapping $F : M \longrightarrow N$ is a local isometry if and only if $E = \bar{E}$, $F = \bar{F}$, $G = \bar{G}$ for each patch x in M.	Page 266, Lemma 4.5
(vi)	Local isometries are conformal mappings with $\lambda(p) = 1$.	Page 268, line −2

(vii) Each of the following conditions is equivalent to F being a local isometry:

- (a) F_* preserves dot products;
- (b) F_* preserves lengths;
- (c) F_* preserves tangent frames;
- (d) F_* preserves $\|v\|$, $\|w\|$ and $v \cdot w$ for some pair of linearly independent tangent vectors v and w in $T_p(M)$ for each p.

Text, page 19, Theorem VI.4.A and Page 269, Exercise 1

Techniques

(i) Recognizing isometries of surfaces and local isometries.

Page 264, Definition 4.2 and Page 265, Definition 4.4

(ii) Finding local isometries using Result (v).

Page 266, Lemma 4.5

Exercises

Technique (i)

2. Let C be the catenic cylinder, which is the image of the patch

$$x : (u, v) \longrightarrow (\sinh^{-1} u, (1 + u^2)^{\frac{1}{2}}, v) \qquad (u, v) \in E^2.$$

That is, C is a cylinder whose cross-section is a catenary (see Fig. 5.41 on page 239). Show that C is isometric to E^2 by showing that $\underset{\sim}{x}$ is an isometry of surfaces. (The derivative of $u \longmapsto \sinh^{-1} u$ is $u \longmapsto (1 + u^2)^{-\frac{1}{2}}$.)

Technique (ii)

3. Page 270, Exercise 5(a). You may find it helpful to look at Exercise V.5.11 on page 231, where the tangent surface of a curve was defined.

Theory Exercise

4. Prove that, for any diffeomorphism (that is, mapping with an inverse mapping) F of surfaces, $(F^{-1})_* = F_*^{-1}$. Deduce that the relation "is isometric to" is an equivalence relation on the class of all surfaces in E^3.

Optional Section — Conformal Mappings

O'Neill has barely introduced conformal mappings. They are not needed again in this course, but they are relevant to some other parts of Mathematics (M332 Complex Analysis, for example). If you would like to gain a deeper understanding of conformal mappings, work through the following exercises.

5. Write down conditions, in terms of $\lambda(p)$, analogous to (a), (b), (c), (c′), (d) of Theorem VI.4.A which are logically equivalent and which F_{*p} must satisfy for all p if F is to be conformal. For example, (b) becomes the definition of conformal:

   (b) For all v in $T_p(M)$

   $$\|F_*(v)\| = \lambda(p)\,\|v\|\,.$$

6. Page 270, Exercise 8. (ϑ is defined to be an angle between v and w if $\cos\vartheta = v \cdot w/\,\|v\|\,\|w\|$.)

7. Page 271, Exercise 14. Apply condition (d) of Exercise 5 to x_u, x_v for suitable patches x. You will need to use two patches.

   (a) All of Σ_0 except one semicircle from the north pole to the south pole is covered by the modified geographical patch

   $$x(u, v) = (\cos v \cos u, \cos v \sin u, 1 + \sin v), \qquad u \in (-\pi, \pi),\ v \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}).$$

   This patch x, and the composite $P(x) = y$ (where P is stereographic projection) are given on page 162. The partial velocities x_u, x_v, y_u, y_v were computed in the solution to Exercise 3 on page 53 of the Text, Part IV. A similar patch, changing the domain of u to $(0, 2\pi)$, covers all of the missing semicircle except the south pole: the details of the computation are obviously the same and may be omitted.

   (b) The south pole is covered by the Monge patch

   $$x(u, v) = (u, v, 1 - (1 - u^2 - v^2)^{\frac{1}{2}}), \qquad u^2 + v^2 < 1.$$

   The values of the partial velocities of this patch x and the composite P(x) at the origin were computed in the solution to Exercise 3 on page 53 of the Text, Part IV.

Solutions

1. *(c′) ⇒ (d)* Let $v_1 = e_1$, $v_2 = e_2$, where e_1, e_2 is the particular orthonormal basis for V specified in (c′). Then v_1, v_2 are linearly independent, and

   $$\|T(v_1)\| = \|T(e_1)\| = 1 = \|e_1\| = \|v_1\|$$

   $$\|T(v_2)\| = \|T(e_2)\| = 1 = \|e_2\| = \|v_2\|$$

   $$T(v_1) \cdot T(v_2) = T(e_1) \cdot T(e_2) = 0 = e_1 \cdot e_2 = v_1 \cdot v_2$$

   (using (c′)).

   (d) ⇒ (b) Let v_1, v_2 be the particular vectors specified in (d). These are linearly independent and so span V, and thus any vector v in V can be written as a linear combination of v_1, v_2:

   $$v = av_1 + bv_2$$

   where a, b are real numbers. T is a linear transformation so

   $$\begin{aligned} T(v) &= T(av_1 + bv_2) \\ &= aT(v_1) + bT(v_2). \end{aligned}$$

Thus

$$\begin{aligned}\|T(v)\|^2 &= \|aT(v_1)+bT(v_2)\|^2\\ &= (aT(v_1)+bT(v_2))\cdot(aT(v_1)+bT(v_2))\\ &= a^2T(v_1)\cdot T(v_1)+2abT(v_1)\cdot T(v_2)+b^2T(v_2)\cdot T(v_2)\\ &= a^2\|T(v_1)\|^2+2abT(v_1)\cdot T(v_2)+b^2\|T(v_2)\|^2\\ &= a^2\|v_1\|^2+2abv_1\cdot v_2+b^2\|v_2\|^2 \qquad \text{(by (d))}\\ &= (av_1+bv_2)\cdot(av_1+bv_2)\\ &= \|av_1+bv_2\|^2\\ &= \|v\|^2.\end{aligned}$$

2. The patch $x : E^2 \longrightarrow C$ is onto by definition of C. Moreover

$$x : (u, v) \longmapsto (\sinh^{-1}u, (1+u^2)^{\frac{1}{2}}, v) \qquad (u, v) \in E^2$$

is one-to-one because the functions

$$u \longmapsto \sinh^{-1}u \qquad (u \in R)$$

$$v \longmapsto v \qquad (v \in R)$$

are both one-to-one. Thus we have to show just that x_* preserves dot products of tangent vectors. Following the method of Example VI.4.6(1), it suffices to show that E = G = 1, F = 0.

$$x_u = \left(\frac{1}{(1+u^2)^{\frac{1}{2}}},\ \frac{u}{(1+u^2)^{\frac{1}{2}}},\ 0\right)$$

$$x_v = (0,\ 0,\ 1)$$

so $E = x_u \cdot x_u = \dfrac{1+u^2}{1+u^2} = 1, \quad F = x_u \cdot x_v = 0, \quad G = x_v \cdot x_v = 1,$

and so x is an isometry.

3. Page 270, Exercise 5(a).

The tangent surface A to α is the image of the patch

$$x : (u, v) \longmapsto \alpha(u) + vT(u)$$

where T(u) is the vector such that

$$\alpha'(u) = T(u)_{\alpha(u)}.$$

The tangent surface B to β is parametrized by

$$y : (u, v) \longmapsto \beta(u) + v\overline{T}(u)$$

where

$$\beta'(u) = \overline{T}(u)_{\beta(u)}.$$

An obvious mapping from A to B is F, where F is defined by F(x) = y. By Lemma VI.4.5, F is a local isometry if $E = \overline{E}$, $F = \overline{F}$, $G = \overline{G}$. Now

$$\begin{aligned}x_u &= \alpha'(u) + vT'(u)_{\alpha(u)}\\ &= (T(u) + v\kappa(u)N(u))_{\alpha(u)}\end{aligned}$$

where $N(u)_{\alpha(u)}$ is the principal normal to α at $\alpha(u)$, and

$$x_v = T(u)_{\alpha(u)}:$$

thus $E = x_u \cdot x_u = T(u) \cdot T(u) + 2v\kappa(u)T(u) \cdot N(u) + (v\kappa(u))^2 N(u) \cdot N(u)$

$$= 1 + 0 + (v\kappa(u))^2;$$

$$F = x_u \cdot x_v = T(u) \cdot T(u) + v\kappa(u)T(u) \cdot N(u)$$

$$= 1;$$

$$G = x_v \cdot x_v = T(u) \cdot T(u) = 1.$$

Since β has the same curvature function κ, a similar argument shows that

$$\bar{E} = 1 + (v\kappa(u))^2$$

$$\bar{F} = 1$$

$$\bar{G} = 1.$$

Hence $E = \bar{E}$, $F = \bar{F}$, $G = \bar{G}$, and so F is an isometry.

4. By the definition of inverse mapping,

$$FF^{-1} = F^{-1} F = I.$$

Thus the chain rule gives

$$F_*(F^{-1})_* = (F^{-1})_* F_* = I_*.$$

What is I_*? Well, I_* is defined so that

$$I_*(\alpha') = (I(\alpha))'$$

for any tangent vector α' tangent to M. I is the identity map on M, and so $(I(\alpha))' = \alpha'$ and thus $I_*(\alpha') = \alpha'$: in other words I_* is the identity map on the tangent spaces to M. Thus $F_*(F^{-1})_*$ and $(F^{-1})_* F_*$ are both the identity map on the tangent spaces to M, and so $(F^{-1})_*$ is the inverse of F_*; that is,

$$(F^{-1})_* = (F_*)^{-1}.$$

Now, to show that the relation "is isometric to" is an equivalence relation we must show that it is (a) reflexive, (b) symmetric, and (c) transitive.

(a) If M is a surface in E^3, the identity map $I : M \longrightarrow M$ is clearly an isometry, so M is isometric to itself. Thus "is isometric to" is reflexive.

(b) If $F : M \longrightarrow N$ is an isometry of surfaces, we saw on page 265 that F has an inverse mapping $F^{-1} : N \longrightarrow M$. Since F^{-1} has F as an inverse mapping, F^{-1} must itself be one-to-one and onto. To show that F^{-1} is an isometry we need to show that $(F^{-1})_*$ preserves dot products. We have shown that $(F^{-1})_* = (F_*)^{-1}$. Now F_* preserves dot products; that is, each F_{*p} is an orthogonal transformation. Consequently Theorem VI.4.B tells us that each ${F_{*p}}^{-1}$ is an orthogonal transformation; that is ${F_*}^{-1}$ preserves dot products. Hence $(F^{-1})_*$ preserves dot products and so F^{-1} is an isometry.

(c) Suppose $F : M \longrightarrow N$ and $G : N \longrightarrow S$ are isometries of surfaces. Then F and G are both one-to-one and onto, so $GF : M \longrightarrow S$ is also one-to-one and onto. For tangent vectors v, w tangent to M at the same point,

$$\begin{aligned}((GF)_*v)\cdot((GF)_*w) &= G_*(F_*v)\cdot G_*(F_*w), && \text{by the chain rule,}\\ &= F_*v\cdot F_*w, && \text{because G is an isometry,}\\ &= v\cdot w, && \text{because F is an isometry.}\end{aligned}$$

Thus $GF : M \longrightarrow S$ is an isometry and so "is isometric to" is transitive.

5. (a) For all v_1, v_2 in $T_p(M)$

$$F_*(v_1)\cdot F_*(v_2) = (\lambda(p))^2 v_1 \cdot v_2 .$$

(b) For all v in $T_p(M)$

$$\|F_*(v)\| = \lambda(p)\|v\|.$$

(c) For all frames e_1, e_2 in $T_p(M)$

$$\|F_*(e_1)\| = \|F_*(e_2)\| = \lambda(p)$$

$$F_*(e_1)\cdot F_*(e_2) = 0.$$

(c′) There is a particular frame e_1, e_2 in $T_p(M)$ such that

$$\|F_*(e_1)\| = \|F_*(e_2)\| = \lambda(p)$$

$$F_*(e_1)\cdot F_*(e_2) = 0.$$

(d) There is some pair of linearly independent vectors v_1, v_2 in $T_p(M)$ such that

$$\|F_*(v_1)\| = \lambda(p)\|v_1\|$$

$$\|F_*(v_2)\| = \lambda(p)\|v_2\|$$

$$F_*(v_1)\cdot F_*(v_2) = (\lambda(p))^2 v_1 \cdot v_2 .$$

6. Page 270, Exercise 8.

If ϑ is an angle between v and w in $T_p(M)$ then

$$\cos\vartheta = \frac{v\cdot w}{\|v\|\,\|w\|}\,.$$

If $F : M \longrightarrow N$ is conformal with scale factor λ then

$$\|F_*(v)\| = \lambda(p)\|v\|$$

$$\|F_*(w)\| = \lambda(p)\|w\|,$$

by definition of conformal, and

$$F_*(v)\cdot F_*(w) = (\lambda(p))^2\, v\cdot w$$

by condition (a) in Exercise 5. Hence

$$\begin{aligned}\frac{F_*(v)\cdot F_*(w)}{\|F_*(v)\|\,\|F_*(w)\|} &= \frac{(\lambda(p))^2\, v\cdot w}{\lambda(p)\|v\|\;\lambda(p)\|w\|}\\ &= \frac{v\cdot w}{\|v\|\,\|w\|}\\ &= \cos\vartheta,\end{aligned}$$

and so ϑ is an angle between $F_*(v)$ and $F_*(w)$.

7. Page 271, Exercise 14.

(a) First we show that the mapping

$$P : (p_1, p_2, p_3) \longmapsto \frac{2}{2-p_3}(p_1, p_2)$$

is conformal on the image of the patch

$$x : (u, v) \longmapsto (\cos v \cos u, \cos v \sin u, 1 + \sin v),$$
$$(u \in (-\pi, \pi), v \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})).$$

Let $y = P(x)$. Then, since $P_*(x_u) = y_u$, $P_*(x_v) = y_v$, by condition (d) of Exercise 5 we have to show that

$$y_u \cdot y_u = \lambda^2 x_u \cdot x_u$$
$$y_u \cdot y_v = \lambda^2 x_u \cdot x_v$$
$$y_v \cdot y_v = \lambda^2 x_v \cdot x_v$$

for some λ depending on the point p. Now, we saw on page 53 of Text, Part IV that

$$x_u = (-\cos v \sin u, \cos v \cos u, 0)$$
$$x_v = (-\sin v \cos u, -\sin v \sin u, \cos v)$$
$$y_u = \frac{2\cos v}{1-\sin v}(-\sin u, \cos u)$$
$$y_v = \frac{2}{1-\sin v}(\cos u, \sin u)$$

and so

$$x_u \cdot x_u = \cos^2 v, \quad x_u \cdot x_v = 0, \quad x_v \cdot x_v = 1$$
$$y_u \cdot y_u = \frac{4\cos^2 v}{(1-\sin v)^2}, \quad y_u \cdot y_v = 0, \quad y_v \cdot y_v = \frac{4}{(1-\sin v)^2}.$$

Thus we have the required result with

$$\lambda(x(u, v)) = \frac{2}{1-\sin v}.$$

If $p = x(u, v)$ then $P(p) = \frac{2\cos v}{1-\sin v}(\cos u, \sin u)$, so

$$\begin{aligned} 1 + \frac{\|P(p)\|^2}{4} &= 1 + \frac{\cos^2 v}{(1-\sin v)^2} \\ &= \frac{1 - 2\sin v + \sin^2 v + \cos^2 v}{(1-\sin v)^2} \\ &= \frac{2(1-\sin v)}{(1-\sin v)^2} \\ &= \frac{2}{1-\sin v} \\ &= \lambda(p), \end{aligned}$$

as required.

Now we show that P is conformal at the south pole by using the Monge patch

$$x(u, v) = (u, v, 1 - (1 - u^2 - v^2)^{\frac{1}{2}}), \qquad u^2 + v^2 < 1.$$

Again we let $y = P(x)$. We saw on page 53 of Text, Part IV that, at the south pole,

$$x_u(0, 0) = (1, 0, 0)$$
$$x_v(0, 0) = (0, 1, 0)$$
$$y_u(0, 0) = (1, 0)$$
$$y_v(0, 0) = (0, 1)$$

and so

$$x_u \cdot x_u = 1, \quad x_u \cdot x_v = 0, \quad x_v \cdot x_v = 1$$

$$y_u \cdot y_u = 1, \quad y_u \cdot y_v = 0, \quad y_v \cdot y_v = 1$$

Thus we can take $\lambda(0) = 1$ to satisfy condition (d). Then

$$1 + \frac{\|P(0)\|^2}{4} = 1 + 0 = \lambda(0)$$

and so

$$\lambda(p) = 1 + \frac{\|P(p)\|^2}{4}$$

for all p in Σ_0.

Since we have found a positive value of λ for all p in Σ_0, P is conformal.

VI.5 INTRINSIC GEOMETRY OF SURFACES IN E^3

Introduction

This section follows on directly from Section VI.4. It also uses the function F^*, which was defined in Section IV.5, and the connection form computations from Section VI.2.

The purpose of this section is to prove Gauss' *theorema egregium,* that the Gaussian curvature of a surface in E^3 is left invariant by isometries. This is a very important result, which immediately gives us a simple criterion for telling when two surfaces are not isometric. The proof of the theorem is a little technical: we do not expect you to be able to reproduce it from memory.

READ: Section VI.5 (pages 271-275).

Comment

(i) *Page 272 : statement of Lemma 5.1* This is really saying that ω_{12} is the unique 1-form that satisfies

$$d\theta_1 = \omega_{12} \wedge \theta_2$$

$$d\theta_2 = -\omega_{12} \wedge \theta_1$$

and that ω_{21} is given by

$$\omega_{21} = -\omega_{12}.$$

Supplementary Comments

(i) *Page 272 : proof of Lemma 5.1*

$$\begin{aligned} d\theta_1(E_1, E_2) &= (\omega_{12} \wedge \theta_2)(E_1, E_2), && \text{by the first structural equation,}\\ &= \omega_{12}(E_1)\theta_2(E_2) - \omega_{12}(E_2)\theta_2(E_1), && \text{by the definition of wedge product,}\\ &= \omega_{12}(E_1)\cdot 1 - \omega_{12}(E_2)\cdot 0, && \text{by the definition of } \theta_2,\\ &= \omega_{12}(E_1) \end{aligned}$$

and

$$\begin{aligned} d\theta_2(E_1, E_2) &= (\omega_{21} \wedge \theta_1)(E_1, E_2)\\ &= \omega_{21}(E_1)\theta_1(E_2) - \omega_{21}(E_2)\theta_1(E_1)\\ &= -\omega_{21}(E_2)\\ &= \omega_{12}(E_2). \end{aligned}$$

(ii) *Page 272 : end of Remark 5.2* To show that ω_{12} defined by

$$\omega_{12}(E_i) = d\theta_i(E_1, E_2)$$

satisfies the structural equations

$$d\theta_1 = \omega_{12} \wedge \theta_2$$

$$d\theta_2 = -\omega_{12} \wedge \theta_1$$

it suffices to check these equations for the pair (E_1, E_2) (by Lemma VI.2.1).

Now

$$(\omega_{12} \wedge \theta_2)(E_1, E_2) = \omega_{12}(E_1)\theta_2(E_2) - \omega_{12}(E_2)\theta_2(E_1),$$ by definition of th wedge product,

$$= \omega_{12}(E_1),$$ since $\theta_2(E_i) = \delta_{i2}$,

$$= d\theta_1(E_1, E_2),$$ by the new definition of ω_{12},

so

$$\omega_{12} \wedge \theta_2 = d\theta_1.$$

Similarly

$$(-\omega_{12} \wedge \theta_1)(E_1, E_2) = -\omega_{12}(E_1)\theta_1(E_2) + \omega_{12}(E_2)\theta_1(E_1)$$

$$= \omega_{12}(E_2)$$

$$= d\theta_2(E_1, E_2),$$ by the new definition of ω_{12};

so

$$-\omega_{12} \wedge \theta_1 = d\theta_2.$$

Summary

Definitions

(i)	Tangent frame field	Page 271, line −4
(ii)	Isometric invariant	Page 271, line 3 of the section
(iii)	Intrinsic geometry	

Results

(i)	ω_{12} is the only 1-form that satisfies $d\theta_1 = \omega_{12} \wedge \theta_2$ $d\theta_2 = -\omega_{12} \wedge \theta_1$.	Page 272, Lemma 5.1
(ii)	$\omega_{12}(E_i) = -\omega_{21}(E_i) = d\theta_i(E_1, E_2)$.	Page 272, Remark 5.2
(iii)	If F is an isometry, E_1, E_2 a tangent frame field, and $F_*(E_i) = \overline{E}_i$, then $\theta_i = F^*\overline{\theta}_i$ $\omega_{12} = F^*\overline{\omega}_{12}$.	Page 273, Lemma 5.3
(iv)	Gaussian curvature is an isometric invariant.	Page 273, Theorem 5.4
(v)	Local isometries preserve Gaussian curvature.	Page 274, line −8
(vi)	Geodesics are isometric invariants.	Text, page 31, Exercise 1
(vii)	Shape operators, principal directions, principal curvatures, mean curvature, principal curves, asymptotic curves, and the forms ω_{13}, ω_{23} are not isometric invariants.	Text, page 31, Exercise 2

Techniques

(i)	Deciding which properties of a surface are isometric invariants.	Page 271, line 3 of the section
(ii)	Using a knowledge of isometric invariants to find properties of surfaces isometric to a known surface.	Page 274, line -7
(iii)	Using a knowledge of isometric invariants to show that two surfaces are not isometric.	Page 275, line 2

Exercises

Technique (i)

1. Page 275, Exercise 1. (HINT: Since we have seen, on page 228, that non-constant geodesics are all constant-speed reparametrizations of unit-speed geodesics, and since isometries preserve the speed of curves, it is sufficient to consider the case when α has unit speed and use Exercise VI.1.1.)

2. Use Example VI.4.6 (1) on page 267 to deduce that principal vectors, principal curvatures, mean curvatures, shape operators, principal curves, asymptotic curves and the 1-forms ω_{13} and ω_{23} are not isometric invariants.

3. Page 275, Exercise 3. (If you have done M202 or M332 you will appreciate that the truth of the first sentence can be proved as follows:

   $K : M \longrightarrow R$ is a continuous function. Since M is a connected surface and the continuous image of a connected space is also connected, K(M) must be connected. However, $K(M) \subset R$, and the only connected subsets of R are the intervals.)

Technique (ii)

4. Page 275, Exercise 2. Omit the generalization.

Technique (iii)

5. Page 275, Exercise 4.

Solutions

1. Page 275, Exercise 1.

   We wish to show that if $F : M \longrightarrow N$ is a local isometry and α is a unit-speed geodesic in M then $F(\alpha)$ is a geodesic in N. Following the hint to use Exercise VI.1.1 we choose a tangent frame field E_1, E_2 on a region of M containing α such that E_1 restricted to α is its unit tangent T; that is, the value of the vector field E_1 at the point $\alpha(t)$ is the unit tangent $T(t)$, so that

   $$E_1(\alpha(t)) = T(t) = \alpha'(t)$$

   for all t in the domain of α. Because we have chosen E_1 in this way, part (a) of Exercise VI.1.1 tells us that $\omega_{12}(T) = 0$ because α is a geodesic.

Let $\bar{E}_1 = F_*(E_1)$, $\bar{E}_2 = F_*(E_2)$, $\bar{\alpha} = F(\alpha)$, $\bar{T} = \bar{\alpha}'$. Then, as in Lemma VI.5.3, $\bar{E}_1$, $\bar{E}_2$ is a tangent frame field on N and

$$\omega_{12} = F^*(\bar{\omega}_{12}).$$

Moreover, for t in the domain of α,

$$\begin{aligned}
\bar{E}_1(\bar{\alpha}(t)) &= \bar{E}_1(F(\alpha(t))) \\
&= F_*(E_1(\alpha(t))) \\
&= F_*(\alpha'(t)) \\
&= (F(\alpha))'(t), && \text{by definition of } F_*, \\
&= \bar{\alpha}'(t) \\
&= \bar{T}(t)
\end{aligned}$$

so $\bar{E}_1$ coincides with $\bar{T}$ along $\bar{\alpha}$. Now, $\bar{\alpha}$ is also a unit-speed curve, because isometries preserve the speeds of curves, and we have just shown that $\bar{E}_1$ restricted to $\bar{\alpha}$ coincides with its unit tangent $\bar{T}$. Thus the conditions in Exercise VI.1.1 are satisfied by $\bar{\alpha}$, $\bar{E}_1$, $\bar{E}_2$ and so we see that $\bar{\alpha}$ is a geodesic in N if and only if $\bar{\omega}_{12}(\bar{T}) = 0$. Now

$$\begin{aligned}
\bar{\omega}_{12}(\bar{T}) &= \bar{\omega}_{12}(\bar{\alpha}') \\
&= \bar{\omega}_{12}((F(\alpha))') \\
&= \bar{\omega}_{12}(F_*(\alpha')) \\
&= F^*(\bar{\omega}_{12})(\alpha'), && \text{by definition of } F^*, \\
&= \omega_{12}(\alpha'), && \text{by Lemma VI.5.3,} \\
&= \omega_{12}(T) \\
&= 0, && \text{because } \alpha \text{ is a geodesic.}
\end{aligned}$$

Hence $\bar{\alpha}$ is a geodesic in N.

2. We wish to show that certain functions on E^2 and types of curve in E^2 are not "preserved" when mapped to the cylinder $M : x^2 + y^2 = r^2$ by the local isometry

$$x : (u, v) \longmapsto (r\cos(u/r), r\sin(u/r), v).$$

Since each small enough open set in M is isometric to a region in E^2, this will show that these functions and curves are not isometric invariants.

A function f on E^2 is said to be "preserved by x" if the "corresponding function" $\bar{f}$ on M satisfies

$$f = \bar{f}(x).$$

$$\begin{tikzcd}
E^2 \arrow[rr, "x"] \arrow[dr, "f"'] & & M \arrow[dl, "\bar{f}"] \\
& R &
\end{tikzcd}$$

Firstly take f to be each of the principal curvature functions, k_1 and k_2. The principal curvatures of E^2 are zero everywhere, so

$$k_1(p) = k_2(p) = 0$$

for all p in E^2. The principal curvatures of M are $0, -1/r$ everywhere so

$$\overline{k}_1(q) = 0, \overline{k}_2(q) = -1/r$$

for all q in M : in particular

$$\overline{k}_2(x(p)) = -1/r$$

for all p in E^2. Hence $\overline{k}_2(x)$ is not equal to k_1 or k_2 and so x does not preserve the pair of principal curvatures. It is true that in this case x preserves one of the principal curvatures, but this does not happen in general.

A similar treatment works for the mean curvature function H.

Now

$$H(p) = 0$$

for all p in E^2, and

$$\overline{H}(q) = -1/2r$$

for all q in M, so in particular

$$\overline{H}(x(p)) = -1/2r$$

for all p in E^2. Hence $\overline{H}(x)$ is not equal to H, and so x does not preserve mean curvatures.

Take $E_1 = U_1$, $E_2 = U_2$ on E^2, and $\overline{E}_i = x_*(E_i)$. Then ω_{13} is preserved by x if

$$\omega_{13} = \overline{\omega}_{13}(x_*).$$

$$\begin{tikzcd}
T_p(E^2) \arrow[rr, "x_*"] \arrow[dr, "\omega_{13}"'] & & T_{x(p)}(M) \arrow[dl, "\overline{\omega}_{13}"] \\
& R &
\end{tikzcd}$$

We can find the 1-forms ω_{13}, $\overline{\omega}_{13}$ from Corollary VI.1.5 :

$$S = \omega_{13} E_1 + \omega_{23} E_2$$

$$\overline{S} = \overline{\omega}_{13} \overline{E}_1 + \overline{\omega}_{23} \overline{E}_2.$$

On E^2, S is zero and so $\omega_{13} = \omega_{23} = 0$; on M, $S(\overline{E}_1) = -\overline{E}_1/r$ and so $\overline{\omega}_{13}(\overline{E}_1) = -1/r$. Hence

$$\omega_{13}(E_1) = 0$$

whilst

$$\overline{\omega}_{13}(x_*(E_1)) = \overline{\omega}_{13}(\overline{E}_1) = -1/r.$$

Thus x does not preserve ω_{13}. It is true that in this case x preserves ω_{23}, but as the labelling of E_1, E_2 is arbitrary it is clear that ω_{23} is not preserved in general. Contrast this with Lemma VI.5.3(2), which shows that the 1-form ω_{12} *is* an isometric invariant.

We can use the fact that k_1 and k_2 are not in general preserved by isometries to show that the shape operator S is also not preserved in general. The shape operator would be preserved by an isometry F if

$$\bar{S}(F_*(v)) = F_*(S(v))$$

for all tangent vectors v tangent to the domain surface; that is, if the linear transformations S and $F_*^{-1}\ \bar{S}\ F_*$ were equal at each point p of the domain surface. Now the eigenvalues of S are k_1 and k_2; the eigenvalues of $F_*^{-1}\ \bar{S}\ F_*$ are equal to the eigenvalues of $\bar{S}$, which are $\bar{k}_1$ and $\bar{k}_2$: in general these are not the same as k_1 and k_2 and so S and $F_*^{-1}\ \bar{S}\ F_*$ cannot be equal. Thus the shape operator is not preserved.

Tangent vectors v tangent to E^2 with some special property are said to be "preserved by x" if their images $x_*(v)$ have the same property. We note that x_* is onto because x is regular (see the second paragraph on page 161). E^2 is all-umbilic and so every unit tangent vector v tangent to E^2 is a principal vector. Their images $x_*(v)$ are all the unit tangent vectors tangent to M (because x_* is onto and x is a local isometry) and not all of these are principal vectors as there are only four principal vectors at each point of M. Thus x does not preserve principal vectors.

Curves are treated similarly. Curves α in E^2 with a special property are "preserved by x" if their images $x(\alpha)$ have the same property. All curves α in E^2 are principal curves : since x is onto, Lemma IV.3.1, shows that their images $x(\alpha)$ are all of the curves in M, not all of which are principal, as the principal curves in M are just the rulings and the cross-sectional circles, so x does not preserve principal curves. Similarly x does not preserve asymptotic curves because all curves in E^2 are asymptotic but only the rulings on M are.

3. Page 275, Exercise 3.

Let I_M, I_N be the curvature intervals of M, N respectively. We show equality by showing

$$I_M \subset I_N \text{ and } I_N \subset I_M.$$

We have to show both inclusions, because a local isometry which is not actually an isometry does not have an inverse. Suppose $x \in I_M$. Then there is some point p in M with $K(p) = x$. Since local isometries preserve Gaussian curvature

$$K(p) = \bar{K}(F(p)).$$

Thus $x = \bar{K}(F(p))$. Now $F(p) \in N$ so $x \in I_N$. Arguing thus for each $x \in I_M$, we have $I_M \subset I_N$.

Now suppose $y \in I_N$. Then $y = \bar{K}(q)$ for some point q in N. F is onto, so $q = F(p)$ for some p in M. As above

$$K(p) = \bar{K}(F(p)) = \bar{K}(q) = y$$

and so $y \in I_M$. Thus $I_N \subset I_M$ and so $I_M = I_N$.

There are many examples which show that the converse is false. Here is one : the saddle surface $M : z = xy$ has curvature interval $[-1, 0)$ (see page 215) and the helicoid covered by

$$x(u, v) = (u \cos v, u \sin v, bv)$$

has curvature interval $[-1/b^2, 0)$ (see page 214). If we take the helicoid with $b = 1$ then these two surfaces have the same curvature interval but there is no local isometry from one onto the other. The helicoid has Gaussian curvature -1 at all points with $u = 0$, but the saddle surface has Gaussian curvature -1

only at the origin. Thus if F were a local isometry from the helicoid to the saddle surface it would have to map the entire z axis to the origin. Suppose p is a point on the z axis. Then there would be a small neighbourhood of p in the helicoid on which F was an isometry. Such a neighbourhood would have to include other points q on the z axis. Since p and q would both be mapped to $\mathbf{0}$ by F the intrinsic distance between them would not be preserved, contradicting Theorem VI.4.3.

4. Page 275, Exercise 2.

Example VI.4.6(1) tells us that the patch x from E^2 to the circular cylinder $M : x^2 + y^2 = r^2$ given by

$$x : (u, v) \longmapsto (r\cos(u/r), r\sin(u/r), v)$$

is a local isometry. Hence, by the result of Exercise 1, we know that if α is a geodesic in E^2 then $x(\alpha)$ is a geodesic in M. Moreover, because the restriction of x to small enough regions is an isometry, all geodesics on M are of the form $x(\alpha)$ for some geodesic α on E^2. The non-constant geodesics in E^2 are simply the straight lines (see Example V.5.8(1) on page 228)

$$t \longmapsto p + tq$$

where $p, q \in E^2$, $q \neq \mathbf{0}$. If α is the geodesic

$$\alpha : t \longmapsto (p_1, p_2) + t(q_1, q_2)$$

then

$$x(\alpha) : t \longmapsto x(p_1 + tq_1, p_2 + tq_2)$$

$$= \left(r\cos\left(\frac{p_1 + tq_1}{r}\right), r\sin\left(\frac{p_1 + tq_1}{r}\right), p_2 + tq_2\right).$$

Putting $a = q_1/r$, $b = p_1/r$, $c = q_2$, $d = p_2$ shows that the non-constant geodesics on M are the curves of the form

$$t \longmapsto (r\cos(at + b), r\sin(at + b), ct + d),$$

as found in Example V.5.8(3) on page 229.

5. Page 275, Exercise 4.

We have the following curvature intervals:

sphere	$\{1/r^2\}$	
torus	$[-1/r(R-r), 1/r(R + r)]$	(see page 236)
helicoid	$[-1/b^2, 0)$	(see page 214)
circular cylinder	$\{0\}$	
saddle surface	$[-1, 0)$	(see page 215)

There are no values of r, R, b for which any two of these intervals can be equal except those of the helicoid ($b = 1$) and the saddle surface, which we have already shown (Exercise 3) are not isometric. By the main result of Exercise 3, no two of these surfaces can be isometric.

VI.6 ORTHOGONAL COORDINATES

Introduction

This section follows on from Section VI.1 and VI.2 and develops a simple formula for the Gaussian curvature of a region described by an orthogonal patch.

READ: Section VI.6 (pages 276-279).

Supplementary Comment

(i) *Page 277 : Equation (3)* Since θ_1, θ_2 form a "basis" for the 1-forms so do du and dv. Hence $\omega_{12} = a\,du + b\,dv$. The first structural equations become

$$d\theta_1 = \omega_{12} \wedge \theta_2$$

$$= a\,du \wedge \theta_2, \qquad \text{since } \theta_2, = \sqrt{G}\,dv,$$

and

$$d\theta_2 = \omega_{21} \wedge \theta_1 = -\omega_{12} \wedge \theta_1$$

$$= -b\,dv \wedge \theta_1.$$

Comparison with the equations on lines −13 and −14 gives

$$a = \frac{-(\sqrt{E})_v}{\sqrt{G}}, \quad b = \frac{(\sqrt{G})_u}{\sqrt{E}}$$ and the formula for ω_{12} follows.

Summary

Definitions

(i)	Orthogonal patch	Page 276, line −14
(ii)	Associated frame field	Page 276, Definition 6.1

Results

(i)	$E_1 = x_u/\sqrt{E}$ and $E_2 = x_v/\sqrt{G}$.	Page 277, Equation (1)
(ii)	$\theta_1 = \sqrt{E}\,du$ and $\theta_2 = \sqrt{G}\,dv$.	Page 277, Equation (2)
(iii)	$\omega_{12} = -\frac{(\sqrt{E})_v}{\sqrt{G}}\,du + \frac{(\sqrt{G})_u}{\sqrt{E}}\,dv.$	Page 277, Equation (3)
(iv)	$K = -\frac{1}{\sqrt{(EG)}}\left\{\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u + \left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v\right\}$	Page 278, Lemma 6.3

Technique

(i) Computing θ_1, θ_2, ω_{12}, K for an orthogonal patch by the technique that was used to obtain the above results.

Exercise

Technique (i)

1. Page 279, Exercise 1(a).

Solution

1. Page 279, Exercise 1(a).

$x_u = (\cos v, \sin v, 0)$ $E = 1$

$x_v = (-u \sin v, u \cos v, b)$ $F = 0$

$G = u^2 + b^2$

$$\theta_1 = \sqrt{E}\, du = du, \quad \theta_2 = \sqrt{G}\, dv = (u^2 + b^2)^{\frac{1}{2}} dv$$

$$d\theta_1 = 0, \quad d\theta_2 = \frac{u}{(u^2 + b^2)^{\frac{1}{2}}}\, du\, dv = \frac{-udv}{(u^2 + b^2)^{\frac{1}{2}}} \wedge \theta_1$$

$$\omega_{12} = \frac{-(\sqrt{E})_v}{\sqrt{G}}\, du + \frac{(\sqrt{G})_u}{\sqrt{E}}\, dv = \frac{udv}{(u^2 + b^2)^{\frac{1}{2}}}$$

$$d\omega_{12} = \frac{b^2}{(u^2 + b^2)^{\frac{3}{2}}}\, du\, dv = \frac{b^2}{(u^2 + b^2)^2}\, \theta_1 \wedge \theta_2$$

Hence $K = \dfrac{-b^2}{(u^2 + b^2)^2}$.

VI.8 CONGRUENCE OF SURFACES

Introduction

This section draws on the results of all the previous sections of O'Neill in M334 except Section VI.3.

At last we relate the concept of "isometry of E^3" (Chapter III) with that of "isometry of surfaces" (Sections VI.4 and VI.5). We find that Euclidean isometries do induce isometries of surfaces, but that the converse is not true : isometries of surfaces need to preserve shape operators in order to be realizable as Euclidean isometries. In fact the shape operator (Chapter V) turns out to be the analogue for surfaces of the curvature and torsion functions (Section II.3) for curves, in that it determines the congruence of surfaces precisely.

You should be familiar with Section III.5 before reading this section, as the ideas and results developed here are similar to those (but a little harder!). In particular some details from the proof of Theorem III.5.7 are needed in the proof of Theorem VI.8.3 : you should read this latter proof, but we do not expect you to be able to reproduce it.

READ: Section VI.8 (pages 297-301).

Supplementary Comments

(i) *Page 297 : line −7* The fact that the restriction $F : M \longrightarrow \bar{M}$ is a mapping is a consequence of Comment (i) on page 33 of Text, Part IV.

(ii) *Page 298 : last four lines of proof*

$$\begin{aligned}
F_*(S(v)) &= F_*(-\nabla_v U), && \text{because } S(v) \text{ is defined to be } -\nabla_v U,\\
&= F_*(-U'(0)), && \text{(where } U \text{ is restricted to } \alpha\text{) by the result of Exercise V.1.1,}\\
&= -F_*(U'(0)), && \text{because } F_* \text{ is linear,}\\
&= -(F_*(U))'(0), && \text{because } F_* \text{ preserves derivatives of vector fields (Corollary III.4.1),}\\
&= -\bar{U}'(0), && \text{because } \bar{U} = F_*(U).
\end{aligned}$$

Restricting U to α automatically restricts $\bar{U}$ to $F(\alpha)$, which is a curve in $\bar{M}$ with initial velocity $F_*(v)$: hence the result of Exercise V.1.1 can be used again to give

$$\bar{S}(F_*(v)) = -\bar{U}'(0)$$

and so

$$\bar{S}(F_*(v)) = F_*(S(v)).$$

(iii) *Page 299 : proof of Lemma 8.2* The precise form of Corollary VI.1.5 that is used is

$$S(v)\cdot E_i(p) = \omega_{i3}(v)$$

and the similar expression for $\bar{M}$.

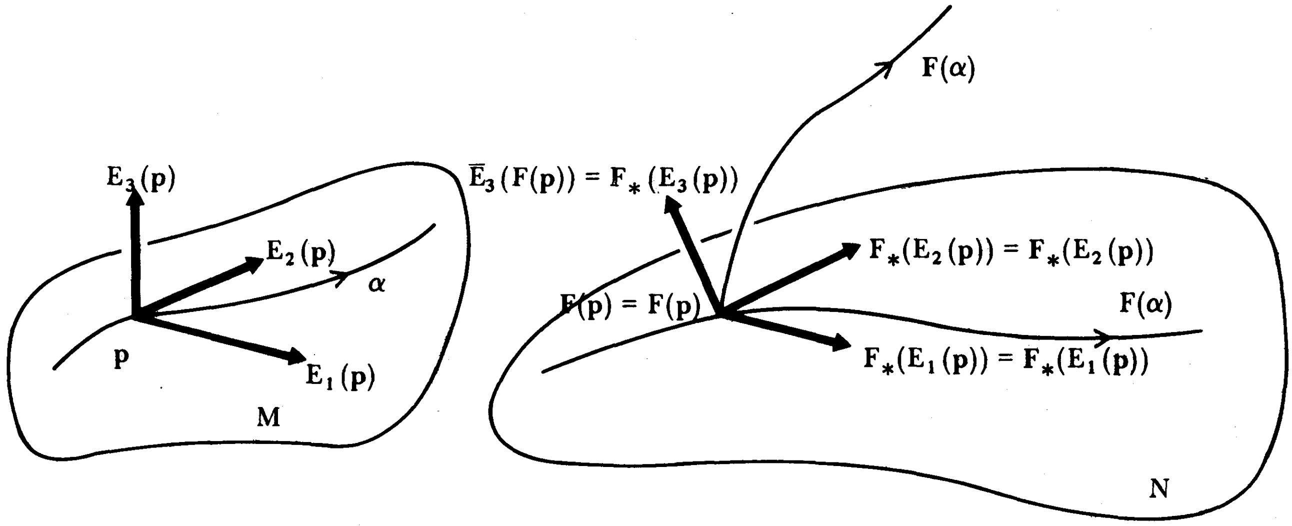
(iv) *Page 299 : lines −7 to −2* Let $E_1(p)$, $E_2(p)$ be a tangent frame to M at p. Then $E_1(p)$, $E_2(p)$, $E_3(p)$ is a frame at p. Moreover $F_*(E_1(p))$, $F_*(E_2(p))$ is a tangent frame to $\overline{M}$ at F(p), and so $F_*(E_1(p))$, $F_*(E_2(p))$, $\overline{E}_3(F(p))$ is a frame at F(p). By Theorem III.2.3, there is a Euclidean isometry F such that

$$\begin{aligned} F(p) &= F(p) \\ F_*(E_1(p)) &= F_*(E_1(p)) \\ F_*(E_2(p)) &= F_*(E_2(p)) \\ F_*(E_3(p)) &= \overline{E}_3(F(p)). \end{aligned}$$

Now if $v \in T_p(M)$, $v = aE_1(p) + bE_2(p)$ for some real numbers a and b, so

$$\begin{aligned} F_*(v) &= F_*(aE_1(p) + bE_2(p)) \\ &= aF_*(E_1(p)) + bF_*(E_2(p)) \\ &= aF_*(E_1(p)) + bF_*(E_1(p)) \\ &= F_*(aE_1(p) + bE_2(p)) \\ &= F_*(v). \end{aligned}$$

(v) *Page 300 : lines 1, 2* We have F and F agreeing at p, and want to show that F(q) = F(q) for all q in M. For a particular q we choose a curve α in M from p to q. Then $F(\alpha)$ is a curve in N and $F(\alpha)$ is a curve in E^3.


$E_3(p)$
$E_2(p)$
α
p
$E_1(p)$
M
$F(\alpha)$
$\overline{E}_3(F(p)) = F_*(E_3(p))$
$F_*(E_2(p)) = F_*(E_2(p))$
$F(p) = F(p)$
$F(\alpha)$
$F_*(E_1(p)) = F_*(E_1(p))$
N

It suffices to show that $F(\alpha) = F(\alpha)$, for then F(q) = F(q) because $q = \alpha(t)$ for some t.

(vi) *Page 301 : last four lines of proof* We compare the equations (C1) and (C2):

$$\begin{aligned} (F_*E_i)' \cdot F_*E_j &= \overline{E}_i' \cdot \overline{E}_j \\ F(\alpha)' \cdot F_*E_i &= \overline{\alpha}' \cdot \overline{E}_i \end{aligned}$$

where $\{F_*E_i\}$, $\{\overline{E}_i\}$ are frame fields on $F(\alpha)$, $\overline{\alpha}$ respectively, with the conditions for Theorem III.5.7:

$$\begin{aligned} E_i' \cdot E_j &= F_i' \cdot F_j \\ \alpha' \cdot E_i &= \beta' \cdot F_i \end{aligned}$$

where $\{E_i\}$, $\{F_i\}$ are frame fields on α, β respectively. We see that $F(\alpha)$, $\bar{\alpha}$, $\{F_*E_i\}$, $\{\bar{E}_i\}$ satisfy all the conditions for Theorem III.5.7, and so we can draw the conclusion of that theorem, namely that $F(\alpha)$ and $\bar{\alpha}$ are congruent. We may assume that $\alpha(0) = p$. Then the details at the beginning of the proof of Theorem III.5.7 show that the Euclidean isometry G carrying $F(\alpha)$ to $\bar{\alpha}$ is the unique Euclidean isometry which takes F_*E_i at $F(\alpha(0))$ to $\bar{E}_i$ at $\bar{\alpha}(0)$ for $i = 1, 2, 3$. Now $\alpha(0) = p$, so

$$F(\alpha(0)) = F(p)$$

and

$$\bar{\alpha}(0) = F(\alpha(0)) = F(p).$$

Moreover, for $i = 1, 2, 3$, $(F_*(E_i))(F(p)) = F_*(E_i(p))$;

for $i = 1, 2$, $F_*(E_i(p)) = F_*(E_i(p))$, by the choice of F on page 299,

$= \bar{E}_i(F(p))$, by the choice of $\bar{E}_i$ on page 300,

and $F_*(E_3(p)) = \bar{E}_3(F(p))$, by the choice of F on page 299.

Thus G leaves invariant the point $F(p)$, and the frame $F_*(E_i(p))$ at that point, and so G, which is unique, must be the identity. Hence

$$F(\alpha) = \bar{\alpha} = G(F(\alpha)) = F(\alpha).$$

Summary

Definition

(i)	Congruent surfaces	Page 297, line 1 of the section

Example

(i)	The saddle surfaces $M : z = xy$ and $M : z = (x^2 - y^2)/2$ are congruent.	Page 297, line 5 of the section

Results

(i)	The restriction of a Euclidean isometry to an oriented surface in E^3 is an isometry of surfaces which preserves shape operators, up to the usual ambiguity in sign (congruent surfaces are isometric).	Page 297, Theorem 8.1
(ii)	If F is an isometry of oriented surfaces which preserves the shape operators defined in terms of the orienting unit normals E_3 and $\bar{E}_3$ and which takes the tangent frame E_1, E_2 to the tangent frame $\bar{E}_1, \bar{E}_2$, then $F^*(\bar{\omega}_{ij}) = \omega_{ij}, (1 \leqslant i, j \leqslant 3)$.	Page 299, Lemma 8.2
(iii)	If F is an isometry of oriented surfaces that preserves shape operators then F is the restriction to the surface concerned of a Euclidean isometry.	Page 299, Theorem 8.3
(iv)	Euclidean isometries preserve mean, Gaussian and principal curvatures, principal vectors, umbilics, asymptotic and principal curves and geodesics.	Text, Page 41, Exercise 1

Techniques

(i)	Finding properties of surfaces invariant under Euclidean isometries.	Page 297, Theorem 8.1 and Page 299, Lemma 8.2
(ii)	Finding Euclidean isometries between surfaces.	Page 299, proof of Theorem 8.3
(iii)	Using Result (iii).	Page 299, Theorem 8.3
(iv)	Using results about congruence of curves to obtain results about congruence of connected surfaces.	

Exercises

Technique (i)

1. Page 301, Exercise 3. (HINT : Choose a unit normal (and hence shape operator) on each of M, N and use Theorem VI.8.1.)

Technique (ii)

2. Page 303, Exercise 9. (HINT : First show that $F(0) = 0$ in each case, using Theorem VI.8.1 and the theorema egregium. Then use Result (iv) and Exercise V.2.1(b)). A Euclidean symmetry is defined in Exercise 7.

Technique (iii)

3. Page 302, Exercise 4.

Technique (iv)

4. Page 301, Exercise 2. The tangent surface is defined in Exercise V.5.11(b). (HINT : Use Theorem III.5.3 to show that there is a Euclidean isometry F with $F(\alpha) = \beta$; write $F = T_aC$ in the usual way and show that

$$F(\alpha(u) + vT(u)) = \beta(u) + v\bar{T}(u),$$

where $T, \bar{T}$ are the unit tangent vector fields on α, β respectively.)

Solutions

1. Page 301, Exercise 3.

Since $F|M$ is, by Theorem VI.8.1, an isometry of surfaces, $F|M$ preserves Gaussian curvature and geodesics as these are preserved by all isometrics of surfaces.

Writing F for $F|M$, we know that F preserves shape operators, that is

$$F_*(S(v)) = \bar{S}(F_*(v))$$

for all v tangent to M. If $S(v) = kv$ then

$$\bar{S}(F_*(v)) = F_*(S(v)) = F_*(kv) = kF_*(v)$$

and so F takes eigenvectors and eigenvalues of S into eigenvectors and eigenvalues of $\bar{S}$; that is, F preserves principal vectors (and hence principal directions) and principal curvatures. The mean curvature is the arithmetic mean of the principal curvatures, and so this is also preserved. Moreover, if p is umbilic then all tangent vectors in $T_p(M)$ are eigenvectors so all tangent vectors in $T_{F(p)}(N)$ are eigenvectors, so $F(p)$ is also umbilic.

Now let α be a curve in M. If α is asymptotic then $S(\alpha') = 0$ so

$$\bar{S}(F(\alpha)') = \bar{S}(F_*(\alpha')) = F_*(S(\alpha')) = F_*(0) = 0$$

and so $F(\alpha)$ is asymptotic also. If α is principal then α' is always an eigenvector of S so $F(\alpha)' = F_*(\alpha')$ is always an eigenvector of $\bar{S}$, so $F(\alpha)$ is principal also.

We saw in Section VI.5 that only Gaussian curvature and geodesics are preserved by arbitrary isometries.

2. Page 303, Exercise 9.

Suppose F is a Euclidean symmetry of M; that is, F is a Euclidean isometry and $F(M) = M$. Then by Theorem VI.8.1 $F|M$ is an isometry of surfaces, and so $F|M$ preserves Gaussian curvature by the theorema egregium.

$M : z = xy$, and so every point p on M has the form (p_1, p_2, p_1p_2). We have seen before (page 215) that the Gaussian curvature at the point $(p_1, p_2, p_1 p_2)$ of M is $-(1 + p_1^2 + p_2^2)^{-2}$, which is -1 only at **0**, and so F must leave the point **0** invariant. Now by Lemma III.1.6 F is an orthogonal transformation.

Clearly F cannot be just any orthogonal transformation. Result (iv) shows that F must preserve the principal vectors at **0**. We saw in Exercise V.2.1 that these are

$$\pm(\mathbf{u}_1 \pm \mathbf{u}_2)/\sqrt{2}$$

where $\mathbf{u}_1 = U_1(0)$, $\mathbf{u}_2 = U_2(0)$. The vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ make angles of $\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$ with the principal vectors. The derivative map F_* preserves dot products (and hence angles) so $F_*(\mathbf{u}_1)$ and $F_*(\mathbf{u}_2)$ are orthogonal unit vectors making angles of $\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$ with the principal vectors. This gives the eight possibilities:

$$F_*(\mathbf{u}_1) = \pm\,\mathbf{u}_i,\ F_*(\mathbf{u}_2) = \pm\,\mathbf{u}_j,\ (\{i, j\} = \{1, 2\}).$$

Since F is an orthogonal transformation, F and F_* have the same matrix C representing them (Theorem III.2.1). We have shown that the first two columns of C must be one of:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}$$

and since C is orthogonal this means that the third column must be

$$\begin{pmatrix} 0 \\ 0 \\ \varepsilon \end{pmatrix} \quad \text{where } \varepsilon = \pm 1.$$

We test the 16 possibilities we now have by using the fact that F(M) = M only if

$$C\begin{pmatrix} x \\ y \\ xy \end{pmatrix} \quad \text{is of the form} \quad \begin{pmatrix} u \\ v \\ uv \end{pmatrix}$$

Since

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}\begin{pmatrix} x \\ y \\ xy \end{pmatrix} = \begin{pmatrix} x \\ y \\ \varepsilon xy \end{pmatrix}$$

we can have $\varepsilon = 1$ but not $\varepsilon = -1$ in this case;
Since

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}\begin{pmatrix} x \\ y \\ xy \end{pmatrix} = \begin{pmatrix} -x \\ y \\ \varepsilon xy \end{pmatrix}$$

we can have $\varepsilon = -1$ but not $\varepsilon = 1$ in this case : working similarly we find that the eight Euclidean symmetries of M are the orthogonal transformations with matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

F carries M *onto* M in each case because the point (x, y, xy) is the image of one of the eight points (±x, ±y, (±x)(±y)), (±y, ±x, (±y)(±x)).

3. Page 302, Exercise 4.

Suppose Σ has radius r, Σ' has radius R. If $p \in \Sigma, K(p) = 1/r^2$ and $\bar{K}(F(p)) = 1/R^2$. Since F is an isometry of surfaces F preserves Gaussian curvature and so $r = R$.

Orient Σ and Σ' by their outward normals. Then (see page 192) $S(v) = -v/r$ for tangent vectors v tangent to Σ and $\bar{S}(w) = -w/r$ for tangent vectors w tangent to Σ'. In particular if we take $w = F_*(v)$ we obtain

$$F_*(S(v)) = F_*(-v/r) = -F_*(v)/r = \bar{S}(F_*(v)).$$

Thus F is an isometry of surfaces preserving shape operators, and so by Theorem VI.8.3 there is a Euclidean isometry F such that $F = F|M$.

4. Page 301, Exercise 2.

We have the surfaces M_α and M_β parametrized by

$$x : (u, v) \longmapsto \alpha(u) + vT(u)$$
$$y : (u, v) \longmapsto \beta(u) + v\bar{T}(u)$$

respectively, where

$$\alpha'(u) = T(u)_{\alpha(u)}$$
$$\beta'(u) = \bar{T}(u)_{\beta(u)}.$$

Since $\kappa_\alpha = \kappa_\beta > 0$ and $\tau_\alpha = \tau_\beta$, Theorem III.5.3 shows that there is a Euclidean isometry F such that $F(\alpha) = \beta$. We aim to show that $F(M_\alpha) = M_\beta$ to demonstrate that M_α is congruent to M_β, and we can do this by showing

$$F(\alpha(u) + vT(u)) = \beta(u) + v\bar{T}(u).$$

Write $F = T_aC$, where T_a is translation by a and C is an orthogonal transformation, as in Chapter III. Applying Theorem III.2.1 to $\alpha'(u)$ gives

$$F_*(\alpha'(u)) = F_*(T(u)_{\alpha(u)}) = CT(u)_{F(\alpha(u))}.$$

However, $F(\alpha) = \beta$ and $F_*(\alpha') = (F(\alpha))' = \beta'$, so

$$CT(u)_{\beta(u)} = \beta'(u) = \bar{T}(u)_{\beta(u)}$$

and so

$$CT(u) = \bar{T}(u).$$

Now

$$\begin{aligned} F(\alpha(u) + vT(u)) &= C(\alpha(u) + vT(u)) + a \\ &= C\alpha(u) + vCT(u) + a, \qquad \text{because C is linear,} \\ &= (C\alpha(u) + a) + vCT(u) \\ &= F(\alpha(u)) + vCT(u) \\ &= \beta(u) + v\bar{T}(u). \end{aligned}$$

SUMMARY AND LOOK-AHEAD

You have now completed all those sections of O'Neill that are a part of M334. O'Neill gives a short summary on page 303, which you may like to read, although the best summary of this course is probably given by the structure diagrams of Part 0, which should now make more sense to you.

There are, however, still one Chapter and several isolated sections of O'Neill which you have so far not had to read but may someday want to.

To begin with there are Sections IV.6 and VI.7, which deal with integration of forms on a surface: this ties in with the work on Green's theorem in M321.

In Section IV.8 O'Neill tries to give a more general approach to the theory of surfaces which underlies the material of Chapter VII on Riemannian geometry. The approach of Chapter VII is typical of generalizations – theorems of previous sections now become definitions.

In Section VII.1 Riemannian surfaces are introduced as surfaces that are equipped with an arbitrary dot product on their tangent spaces, though in all the examples dealt with the tangent spaces and dot products are derived directly from the usual ones. O'Neill assumes that vector fields, 1-forms, 2-forms all exist on such surfaces and that all the results of the exterior calculus still hold. For instance, he assumes the existence of frame fields and dual forms and then uses the analogue of the structure equations $d\theta_1 = \omega_{12} \wedge \theta_2$ and $d\omega_2 = \omega_{21} \wedge \theta_1$ to define the connection form ω_{12}.

In Section VII.2 the Gaussian curvature is defined using the second structural equation

$$d\omega_{12} = -K\,\theta_1 \wedge \theta_2$$

and it is calculated for several important examples such as the stereographic plane and the hyperbolic plane.

Working backwards in Section VII.3 he shows that it is possible to construct a unique covariant derivative satisfying the connection equations

$$\nabla_V E_1 = \omega_{12}(V)E_2, \qquad \nabla_V E_2 = \omega_{21}(V)E_1,$$

though when we are dealing with a surface in E^3 this turns out to be the restriction of the usual covariant derivative to the tangent planes in question.

This covariant derivative is then used to work backwards to a definition of derivative of a vector field along a curve, and in Section VII.4 this is used to investigate geodesics on a Riemannian surface.

In Sections VII.5 and VII.6 the geodesics are shown to minimize, at least locally, the distance between points, and it is shown how measurement of geodesic distance can be used by "inhabitants" in a surface to determine its Gaussian curvature.

In the final sections of Chapter VII a relationship is found between the Gaussian curvature of a surface, a local property, and the Euler-Poincaré characteristic, a global property, which provides a link between differential geometry and the further disciplines of differential and algebraic topology.

FURTHER EXERCISES AND SOLUTIONS

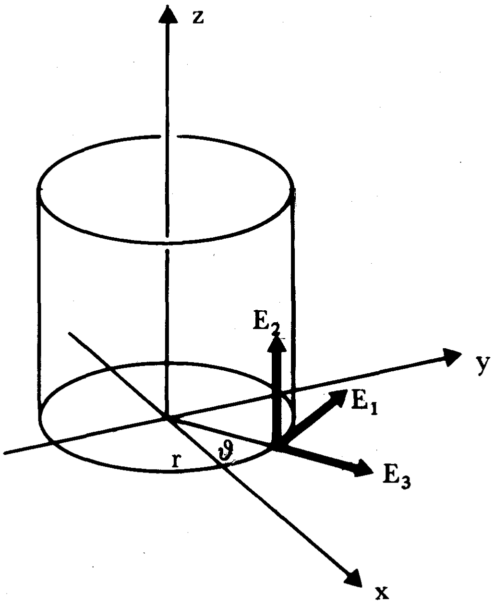
Section VI.1

Technique Exercise

Page 251, Exercise 5.

Solution

5.



$E_1 = -\sin\vartheta U_1 + \cos\vartheta U_2, E_2 = U_3, \quad E_3 = \cos\vartheta U_1 + \sin\vartheta U_2$

$\theta_1 = r d\vartheta, \quad \theta_2 = dz$ by page 96, Exercise 4.

$$\omega = \begin{pmatrix} 0 & 0 & -d\vartheta \\ 0 & 0 & 0 \\ d\vartheta & 0 & 0 \end{pmatrix}$$ by page 90, line 2 and a suitable permutation.

$$S(v) = -d\vartheta(v)E_1 = -\frac{1}{r}\theta_1(v)E_1$$

$$S = \begin{pmatrix} -\frac{1}{r} & 0 \\ 0 & 0 \end{pmatrix}$$

Section VI.2

Technique Exercise

Page 255, Exercise 1.

Solution

1. $E_1[k_2] = (k_1-k_2)\,\omega_{12}(E_2) = 0$

$$E_2[k_1] = (k_1-k_2)\,\omega_{12}(E_1) = \frac{R\sin\varphi}{r(R+r\cos\varphi)^2}$$

Section VI.3

No further exercises are recommended.

Section VI.4

Technique Exercise

Page 269, Exercise 2.

Other Recommended Exercises

Page 269, Exercise 4. This generalizes Theorem VI.4.3.

Page 270, Exercise 7 (second part). This generalizes Lemma VI.4.5.

Page 270, Exercise 10. This also generalizes Lemma VI.4.5.

Solutions

2(a). $\|F(\alpha')\| = \|\alpha'\|$ for all curves α in M

$\Leftrightarrow \|F_*v\| = \|v\|$ for all vectors v tangent to M.

2(b). $\|F(\alpha')\| = \|\alpha'\|$

$$\Rightarrow L(F(\alpha)) = \int_a^b \|F(\alpha)'\| = \int_a^b \|\alpha'\| = L(\alpha).$$

F preserves the lengths of all sub-curves of α

$$\Rightarrow \int_c^d \|F(\alpha')\| = \int_c^d \|\alpha'\| \quad \text{for all } c, d$$

$\Rightarrow \|F(\alpha')\| = \|\alpha'\|$.

4. If $F : M \longrightarrow \overline{M}$ is a local isometry and $p, q \in M$ then

$$\rho(p, q) \geqslant \rho(F(p), F(q)).$$

Proof: $\rho(p, q) = \text{g.l.b. } \{L(\alpha): \alpha \text{ is a curve segment from } p \text{ to } q \text{ in } M\}$

$= \text{g.l.b. } \{L(F(\alpha)): \alpha \text{ is a curve segment from } p \text{ to } q \text{ in } M\}$

$\geqslant \text{g.l.b. } \{L(\beta): \beta \text{ is a curve segment from } F(p) \text{ to } F(q) \text{ in } \overline{M}\}$

because while every $F(\alpha)$ is a curve segment from $F(p)$ to $F(q)$ in $\overline{M}$, there may be curve segments β from $F(p)$ to $F(q)$ in $\overline{M}$ which are not of the form $F(\alpha)$.

7. $E = (\lambda(u, v))^2 = G$

$F = (\lambda(u, v))^2 \cdot 0 = 0$

10(a). They are mapped to (some of) the parameter curves of $\bar{x}$.

10(b). $x_u \cdot x_u = E$; $\quad \bar{x}_u \cdot \bar{x}_u = (\frac{\partial f}{\partial u}\bar{x}_u(f, g)) \cdot (\frac{\partial f}{\partial u}\bar{x}_u(f, g))$

$$= (\frac{\partial f}{\partial u})^2\,\bar{E}(f, g).$$

10(c). Multiply all the right-hand sides by λ^2, where λ is a function of u, v.

Section VI.5

Technique Exercise

Page 276, Exercise 6. (Ignore the comment in brackets.) Assume the helicoid has $b = 1$.

Solution

6. Parametrize H by

$$x(u, v) = (u\cos v, u\sin v, v)$$

and C by

$$y(s, \vartheta) = (\sinh^{-1} s, (1 + s^2)^{\frac{1}{2}}\cos\vartheta, (1 + s^2)^{\frac{1}{2}}\sin\vartheta).$$

On H, $K(x(u, v)) = -(1 + u^2)^{-2} = -1$ on and only on the axis $u = 0$;

on C, $K(y(s,\vartheta)) = -(1 + s^2)^{-2} = -1$ on and only on the minimal circle $s = 0$.

The ruling $v = v_0$ on H is a unit-speed curve α with Gaussian curvature at $\alpha(u)$ given by $K(\alpha(u)) = -(1+u^2)^{-2}$. Put $F(\alpha) = y(s(u), \vartheta(u))$. Then $K(F(\alpha(u)) = -(1+u^2)^{-2} \Rightarrow s(u) = \pm u$ and so $\|F(\alpha)'\| = 1 \Rightarrow \vartheta'(u) = 0 \Rightarrow \vartheta(u)$ is constant $\Rightarrow F(\alpha)$ is a meridan of C.

Section VI.6

Technique Exercises

Pages 279-280, Exercises 1(b), 1(c), 3,4.

Solutions

1(b). $\theta_1 = (1 + u^2)^{\frac{1}{2}}du \qquad \theta_2 = u\,dv$

$\omega_{12} = dv/(1 + u^2)^{\frac{1}{2}}$

$K = 1/(1 + u^2)^2$

1(c). $E_1 = x_u/(1 + a^2)^{\frac{1}{2}} \qquad E_2 = x_v/u$

$\theta_1 = (1 + a^2)^{\frac{1}{2}}du \qquad \theta_2 = u\,dv$

$\omega_{12} = dv/(1 + a^2)^{\frac{1}{2}} \qquad K = 0$

3. $E_1 = x_u \qquad E_2 = (x_v - \cos\vartheta\, x_u)/\sin\vartheta$

$\theta_1 = du + \cos\vartheta\, dv \qquad \theta_2 = \sin\vartheta\, dv$

$\omega_{12} = -\vartheta_u\, du$

$K = -\vartheta_{uv}/\sin\vartheta$

4(a). $S(v) = \omega_{13}(v)E_1 + \omega_{23}(v)E_2$

$$\frac{\ell}{E}x_u = S(x_u) = \omega_{13}(x_u)\frac{x_u}{\sqrt{E}} + \omega_{23}(x_u)\frac{x_v}{\sqrt{G}}$$

$$\frac{n}{G}x_v = S(x_v) = \omega_{13}(x_v)\frac{x_u}{\sqrt{E}} + \omega_{23}(x_v)\frac{x_v}{\sqrt{G}}$$

$$\omega_{13}(x_u) = \frac{\ell}{\sqrt{E}} \qquad \omega_{13}(x_v) = 0 \qquad \omega_{13} = \frac{\ell}{\sqrt{E}}\,du$$

$$\omega_{23}(x_u) = 0 \qquad \omega_{23}(x_v) = \frac{n}{\sqrt{G}} \qquad \omega_{23} = \frac{n}{\sqrt{G}}\,dv$$

4(b). Use the Codazzi equation

$$d\omega_{13} = \omega_{12} \wedge \omega_{23}.$$

$$d\omega_{13} = \left(\frac{\ell E_v}{2E} - \ell_v\right)\frac{dudv}{\sqrt{E}}$$

$$\omega_{12} \wedge \omega_{23} = \frac{-n(\sqrt{E})_v}{G}\,dudv = -\frac{nE_v}{2G}\;\frac{dudv}{\sqrt{E}}$$

Hence $\ell_v = \frac{1}{2}\left(\frac{\ell}{E} + \frac{\ell}{G}\right)E_v = HE_v$, by Exercise V.4.9.

Using $d\omega_{23} = \omega_{21} \wedge \omega_{13}$ we find similarly that $n_u = HG_u$.

Section VI.8

Technique Exercises

Pages 302-3, Exercises 5, 10.

Other Recommended Exercises

Page 302, Exercises 7, 8. These relate Euclidean isometries of a surface to group theory.

Solutions

5(a). $C(p_1, p_2, p_1p_2) = (-p_1, p_2, -p_1p_2)$.

5(b). M, as the image of F, must be oriented by $-U$.

10. K is maximal at $(\pm\alpha, 0, 0)$ and minimal at $(0, 0, \pm c)$. The Euclidean symmetries are the eight orthogonal transformations

$$\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

7(a). Closure and the inclusion of inverses and the identity are trivial to prove.

7(b). If $F : M \longrightarrow N$ is a Euclidean isometry then

$$G \longmapsto F^{-1} GF$$

is an isomorphism of $S(M)$ onto $S(N)$.

8. By Exercise 7(b) we may assume Σ has centre the origin. Let $F = TC$.

$$\begin{aligned} F(\Sigma) = \Sigma &\Leftrightarrow F(0) = 0 \\ &\Leftrightarrow T = I \\ &\Leftrightarrow F = C. \end{aligned}$$

DIFFERENTIAL GEOMETRY

I Calculus on Euclidean Space

II Frame Fields

III Euclidean Geometry

IV Calculus on a Surface

V Shape Operators

VI Geometry of Surfaces in E^3